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Change analysis of dynamic copula for measuring dependence in multivariate financial data

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Abstract. This paper proposes a new approach to measure the dependence in multivariate financial data. Data in finance and insurance often cover a long time period. Therefore, the economic factors may induce some changes inside the dependence structure. Recently, two methods using copulas have been proposed to analyze such changes. The first approach investigates the changes of copula's parameters. The second one tests the changes of copulas by determining the best copulas using moving windows. In this paper we take into account the non stationarity of the data and analyze: (1) the changes of parameters while the copula family keeps static; (2) the changes of copula family. We propose a series of tests based on conditional copulas and goodness-of-fit (GOF) tests to decide the type of change, and further give the corresponding change analysis. We illustrate our approach with Standard & Poor 500 and Nasdaq indices, and provide dynamic risk measures.

Keywords: Dynamic copula; Goodness-of-Fit test; Change-point; Time-varying parameter; VaR; ES.

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1 Introduction

Determining the dependence between assets is an important domain of research. It is useful for portfolio management, risk assessment, option pricing and hedging. The correlation matrices have been a lot considered to quantify the dependence structure between assets,

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but it is now well known that this kind of approach is only satisfactory when we work inside a Gaussian or an Elliptical framework. The recent work of [Embrechts et al. \(2001\)](#) proposing the concept of copulas to measure dependence between financial data has opened the routes to a very interesting research domain, which has shown its ability to improve the domain of quantitative finance.

In the static one-period situation given by the real-valued random variables X_1, \dots, X_d , the dependence between X_1, \dots, X_d is completely determined by their joint distribution function $H(X_1, \dots, X_d)$. The idea of separating H into two parts, one describing the dependence structure and the other one describing the marginal behavior only, leads to the well known concept of copulas, [Joe \(1997\)](#) and [Nelsen \(1999\)](#). When we adjust copulas on financial data sets, generally we assume strict stationarity all along the considered period, and we use different criteria such as AIC criterion, [Akaike \(1974\)](#), D_2 diagnostic, [Caillault and Guégan \(2005\)](#), to determine the best one.

Furthermore, most of data sets cover a reasonably long time period and economic factors may induce some changes in the dependence structure: we can observe tranquil periods and turmoil periods for instance, and then the notion of strict stationarity fails, [Guégan \(2007\)](#). To take into account this phenomenon, the notion of dynamic copula has also been introduced in risk management by [Dias and Embrechts \(2004\)](#), [Jondeau and Rockinger \(2006\)](#), and [Granger et al. \(2006\)](#). The dynamics are introduced inside the copula's parameters using some time-varying function of predetermined variables. In all these cases the family of the copula remains changeless. Recently, [Caillault and Guégan \(2007\)](#) proposed a new method to take into account the possibility of changes of the copula's family and changes inside the parameters, using moving windows. On a sequence of subsamples, a series of copulas (adjusted with respect to the AIC criterion) are selected. Practically the changes of the copulas appear evident. However, some problems remain opened such as the choice of the width of the moving window or the detection of the change points. These choices influence the accuracy of the results for

the copula's adjustment.

In this paper, we develop a new approach to use the dynamic copula. We proceed in two steps. We test if the copula changes. If not we adjust some dynamics on the parameters of the copula. If the copula changes, we adjust a set of copulas to model the dynamics of the data sets. In order to detect the change type of the copula robustly, we propose a series of nested tests based on conditional copulas, [Anderson \(1969\)](#), [Fermanian \(2005\)](#). Our procedure is as follows. At first, we test whether the copula changes during a considered time period. If the copula seems changeless, we keep the copula and we deal with the changes of copula's parameters. If we detect some changes in the copulas, then we apply the so called binary segmentation procedure to detect the change time and to build a sequence of copulas. If only the copula parameters change, we apply the change-point analysis as in [Csörgő and Horváth \(1997\)](#), [Gombay and Horváth \(1999\)](#) and [Dias and Embrechts \(2004\)](#). In this latter case considering that the change-point tests have less power in the case of "small" changes, we assume that the parameters change according to the time-varying functions of some predetermined variables. We summarize our procedure in [Figure 1](#).

In order to illustrate this new approach, we apply it to Standard & Poor 500 and Nasdaq indices. We study their dynamic dependence and use it for risk management, computing risk measures such as the VaR (Value at Risk) and the ES (Expected Shortfall) measures.

The paper is organized as follows. In [Section 2](#), we review some useful notions and specify the notations. [Section 3](#) presents a series of tests for detecting the copulas' change. [Section 4](#) analyzes the details for every change type, including the change time, the copulas and the change value of the parameter, *etc.* In [section 5](#), we provide some empirical research applying the previous method on two real data sets and we associate their dynamic risk measures. [Section 6](#) concludes.

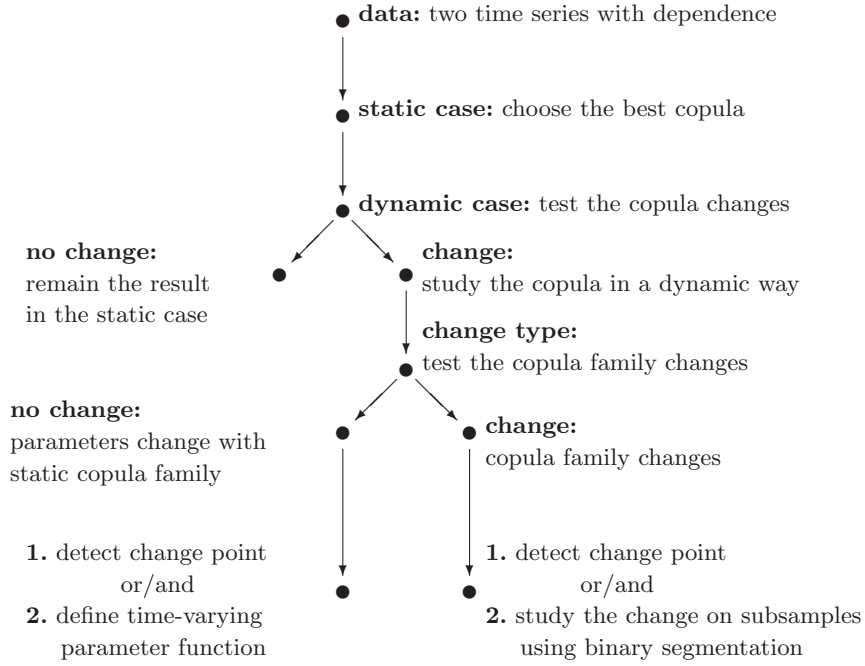


Fig. 1. Change analysis of copula

2 Preliminaries and notations

In order to detect the change of dependence structure, we use conditional copulas. Here we simply recall the definitions and introduce some notations. We specify also some assumptions useful in the following when we apply the Goodness-of-Fit tests.

2.1 Conditional copulas

Following Patton (2006), the conditional copulas are defined as the following.

Definition 1. A d -dimensional conditional copula is a function $C: [0, 1]^d \rightarrow [0, 1]$ such that for some conditioning set \mathcal{F} :

1. For every $\mathbf{u} = (u_1, u_2, \dots, u_d) \in [0, 1]^d$, $C(\mathbf{u}|\mathcal{F}) = 0$ when at least one coordinate of \mathbf{u} is zero, and if all coordinates of \mathbf{u} are 1 except u_k , then $C(\mathbf{u}|\mathcal{F}) = u_k$, $k = 1, \dots, d$.

2. C is d -increasing conditioned on \mathcal{F} ,

The Sklar's theorem (Sklar, 1959) can be extended for conditional distributions and conditional copulas.

Theorem 1. *Let H be a d -dimensional conditional distribution function with continuous margins F_1, F_2, \dots, F_d , and let \mathcal{F} be some conditioning set, then there exists a unique conditional d -copula $C: [0, 1]^d \rightarrow [0, 1]$ such that for all $\mathbf{x} = (x_1, x_2, \dots, x_d)$ in $\overline{\mathbb{R}}^d$,*

$$H(\mathbf{x}|\mathcal{F}) = C(F_1(x_1|\mathcal{F}), F_2(x_2|\mathcal{F}), \dots, F_d(x_d|\mathcal{F})). \quad (1)$$

Conversely, if C is a conditional d -copula and F_1, F_2, \dots, F_d are univariate conditional distribution functions, then the function H defined by Equation (1) is a d -dimensional conditional distribution function with margins F_1, F_2, \dots, F_d .

2.2 Assumptions and Goodness-of-Fit (GOF) tests

Now we specify some useful assumptions for the GOF tests that we use later. For a d -dimensional stationary process with n observations $(\mathbf{X}_n)_{n \in \mathbb{Z}} = \{(X_{i1}, X_{i2}, \dots, X_{in}) : i = 1, 2, \dots, d\}$, let H be its cumulative distribution function. Usually, a GOF test permits to distinguish between two hypotheses. We denote H_0 a known cumulative distribution function, and $\mathcal{H} = \{H_\theta | \theta \in \Theta\}$ a known parametric family of cumulative distribution functions, then the GOF test is:

1. $\mathcal{H}_0 : H = H_0$, against $\mathcal{H}_a : H \neq H_0$, when the null hypothesis is simple; or
2. $\mathcal{H}_0 : H \in \mathcal{H}$, against $\mathcal{H}_a : H \notin \mathcal{H}$ when the null hypothesis is composite.

We specify now some assumptions:

Assumption 1. *Let be K , a probability kernel function on \mathbb{R}^d , twice continuously differentiable, which is the product of d univariate kernels K_i ($i = 1, 2, \dots, d$) with compact supports.*

Assumption 2. *Let be $\mathbf{h}_n = (h_{1n}, h_{2n}, \dots, h_{dn})$ a bandwidth vector, where $h_n = h_{1n} = h_{2n} = \dots = h_{dn}$ such that $h_n \rightarrow 0$, $nh_n^d \rightarrow \infty$,*

$nh_n^{4+d} \rightarrow 0$ and $nh_n^{3+d/2}/(\ln(\ln n))^{3/2} \rightarrow \infty$ as $n \rightarrow \infty$.

Assumption 3. Let be $(\mathbf{X}_n)_{n \in \mathbb{Z}}$, we denote $\varphi_{n-1} = \sigma((X_{1,s}, X_{2,s}, \dots, X_{d,s}) : s \leq n-1)$ the conditional information set available at $n-1$ and $\varphi_{i,n-1} = \sigma(X_{i,s} : s \leq n-1)$ the conditional information set, for the i -th variable, available at $n-1$.

Assumption 4. Let be C_0 the true copula associated to $(\mathbf{X}_n)_{n \in \mathbb{Z}}$. For $\forall \mathbf{u} \in [0, 1]^d$, we denote $c_0 = c_0(\mathbf{u}, \theta)$ its copula density function, and θ the parameter vector. In addition, the first two derivatives of c_0 with respect to \mathbf{u} are assumed to be uniformly continuous on $\Upsilon(\mathbf{u}_j) \times \Upsilon(\theta_0)$, where $\Upsilon(\mathbf{u}_j)$ represents an open neighborhood of the points $(\mathbf{u}_j)_{j=1,2,\dots,m} \in [0, 1]^d$, ($m \in \mathbb{Z}$), $\Upsilon(\theta_0)$ denotes an open neighborhood of θ_0 .

3 Tests for copula's change

In this section we use the conditional copulas to perform a series of specified GOF tests.

3.1 Test to detect the change of copula

Using the previous notations and the notion of the conditional copula, we test the null hypothesis,

$$\mathcal{H}_0^{(1)} : \text{For every } n \in \mathbb{N}, C(\cdot|\varphi_{n-1}) = C_0(\cdot),$$

against

$$\mathcal{H}_a^{(1)} : \text{For some } n \in \mathbb{N}, C(\cdot|\varphi_{n-1}) \neq C_0(\cdot),$$

where C_0 has been introduced before.

In order to apply this test, first we need to build an estimate of the conditional density $c_0(\mathbf{u}_j|\varphi_{n-1})$ at point \mathbf{u}_j . We assume that we observe an n -sample, then its estimate is given by:

$$\hat{c}(\mathbf{u}_j|\varphi_{n-1}) = \frac{1}{nh_n^d} \sum_{i=1}^n K\left(\frac{\mathbf{u}_j - \mathbf{U}_i}{\mathbf{h}_n}\right), \quad (2)$$

where \mathbf{h}_n and h_n are claimed in Assumption 2 and the kernel function K is claimed in Assumption 1. The vector \mathbf{U}_i is such that

$$\mathbf{U}_i = (\hat{F}_1(X_{1,i}), \hat{F}_2(X_{2,i}), \dots, \hat{F}_d(X_{d,i})),$$

$i = 1, 2, \dots, n$, where \hat{F}_l is the empirical l -th marginal cumulative distribution function of $(\mathbf{X}_n)_{n \in \mathbb{Z}}$, for $l = 1, 2, \dots, d$, and

$$\hat{F}_l(X_{l,i}) = \frac{1}{n+1} \sum_{p=1}^n \mathbf{1}_{\{X_{lp} < X_{li}\}}.$$

Now we introduce the test statistics:

$$\mathcal{T} = (nh_n^d) \sum_{j=1}^m \frac{\{\hat{c}(\mathbf{u}_j | \varphi_{n-1}) - c_0(\mathbf{u}_j | \varphi_{n-1})\}^2}{\sigma^2(\mathbf{u}_j)}, \quad (3)$$

where $\sigma(\mathbf{u}_j)$ satisfies:

$$\sigma^2(\mathbf{u}_j) = c_0^2(\mathbf{u}_j | \varphi_{n-1}) \cdot \int K^2.$$

Under the null hypothesis $\mathcal{H}_0^{(1)}$, the statistics \mathcal{T} defined in Equation (3) tends to a Chi-square distribution with m degrees of freedom when $n \rightarrow \infty$, [Fermanian \(2005\)](#). Through this test based on \mathcal{T} , we can detect whether or not the copula changes during a considered time period.

Note that the points $(\mathbf{u}_j)_{j=1,2,\dots,m} \in [0, 1]^d$ are chosen arbitrarily. Clearly, the power of the test \mathcal{T} depends on the choice of the points $(\mathbf{u}_j)_{j=1,2,\dots,m}$, which is a drawback as the choice of cells in the usual GOF Chi-square test. Without a priori, given an integer N , it is always possible to choose a uniform grid of the type $(i_1/N, i_2/N, \dots, i_k/N)$, for every integers $1 \leq i_1, i_2, \dots, i_k \leq N - 1$.

3.2 Test to detect the change type of the copula

If we reject $\mathcal{H}_0^{(1)}$, then we should study the dependence structure inside the d -dimensional vector, in a dynamic way. Thus we test the change type of the copula. Let be $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$ a family of copulas and θ_{n-1} the parameter depending on the past information set

of the process.

Let be the null hypothesis,

$$\mathcal{H}_0^{(2)} : \text{For every } n \in \mathbb{N}, \theta_{n-1} = \theta(\varphi_{n-1}), C(\cdot|\varphi_{n-1}) = C_{\theta_{n-1}} \in \mathcal{C},$$

and the alternative,

$$\mathcal{H}_a^{(2)} : \text{For some } n \in \mathbb{N}, C(\cdot|\varphi_{n-1}) \notin \mathcal{C}.$$

We use the same notations as before and we introduce the statistics associated to this test:

$$\mathcal{R} = (nh_n^d) \sum_{j=1}^m \frac{\{\hat{c}(\mathbf{u}_j|\varphi_{n-1}) - c_{\hat{\theta}_{n-1}}(\mathbf{u}_j|\varphi_{n-1})\}^2}{\hat{\sigma}^2(\mathbf{u}_j)}, \quad (4)$$

where \mathbf{u}_j ($j = 1, 2, \dots, m$) is described in Assumption 4, the σ -algebra φ_{n-1} is introduced in Assumption 3, and $\hat{\theta}_{n-1}$ is the consistent estimator of θ_{n-1} . $c_{\hat{\theta}_{n-1}}(\mathbf{u}_j|\varphi_{n-1})$ denotes the density of the conditional copula $C_{\hat{\theta}_{n-1}}$, and $\hat{c}(\mathbf{u}_j|\varphi_{n-1})$ is the empirical copula density given in Equation (2). Moreover,

$$\hat{\sigma}^2(\mathbf{u}_j) = c_{\hat{\theta}_{n-1}}^2(\mathbf{u}_j|\varphi_{n-1}) \cdot \int K^2.$$

Under the null hypothesis $\mathcal{H}_0^{(2)}$, the statistics \mathcal{R} defined in Equation (4) tends to a Chi-square distribution with m degrees of freedom, when $n \rightarrow \infty$. If we reject $\mathcal{H}_0^{(2)}$, the copula family changes. On the other hand, if we do not reject $\mathcal{H}_0^{(2)}$, the copula family remains static, then we say that only the copula's parameters change. After determining the change type of the copula by testing $\mathcal{H}_0^{(2)}$, we analyze in details the copula's changes.

Note that if we consider the Archimedean copula family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$, the parameter θ can be estimated using the Kendall's tau.

4 Detail analysis for the copula change

According to the test results for the hypotheses $\mathcal{H}_0^{(1)}$ and $\mathcal{H}_0^{(2)}$, we determine the change type of the copula during the time period. Here, we analyze two kinds of changes.

4.1 Detail analysis for the change of copula's family

If we reject $\mathcal{H}_0^{(2)}$, then the copula's family may change. We apply the so called binary segmentation procedure to detect the change point. This procedure proposed by [Vostrikova \(1981\)](#) enables to simultaneously detect the number and the location of the change-points. The procedure can be described as follows. Firstly, we choose the best copula according to the AIC criterion on the whole sample. Then the sample is divided into two subsamples, we choose the best copulas on these two subsamples respectively. If the two best copulas are different from the copula on the whole period, we continue this segmentation procedure, i.e., we again divide each subsample into two parts, and do the same work as in the previous step. Finally, the procedure stops when all the best copulas on the subsamples have been adjusted. Therefore, we get all the change points for the family changes.

4.2 Detail analysis for the change of copula's parameters

If $\mathcal{H}_0^{(2)}$ is not rejected, the copula's family remains changeless. Therefore, we say that only the copula parameters change. Then, we need to deal with the change analysis for the parameters.

To find the change time, we apply the change point technique introduced by [Dias and Embrechts \(2004\)](#). Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be a sequence of independent random vectors in $[0, 1]^d$ with univariate uniformly distributed margins and copulas $C(\mathbf{u}; \theta_1, \eta_1), \dots, C(\mathbf{u}; \theta_n, \eta_n)$, respectively, where θ_i and η_i represent the dynamic and the static copula parameters satisfying $\theta_i \in \Theta^{(1)} \subseteq \mathbb{R}^p$ and $\eta_i \in \Theta^{(2)} \subseteq \mathbb{R}^q$. We test the null hypothesis

$$\mathcal{H}_0^{(3)} : \theta_1 = \theta_2 = \dots = \theta_n \quad \text{and} \quad \eta_1 = \eta_2 = \dots = \eta_n$$

against

$$\mathcal{H}_a^{(3)} : \theta_1 = \dots = \theta_{k^*} \neq \theta_{k^*+1} = \dots = \theta_n \quad \text{and} \quad \eta_1 = \eta_2 = \dots = \eta_n.$$

Here k^* is the location or time of the change-point if we reject the null hypothesis. The hypotheses are tested through the generalized

likelihood ratio, that is, the null hypothesis would be rejected for small values of the likelihood ratio:

$$\Lambda_k = \frac{\sup_{(\theta, \eta) \in \Theta^{(1)} \times \Theta^{(2)}} \prod_{1 \leq i \leq n} c(\mathbf{u}_i; \theta, \eta)}{\sup_{(\theta, \theta', \eta) \in \Theta^{(1)} \times \Theta^{(1)} \times \Theta^{(2)}} \prod_{1 \leq i \leq k} c(\mathbf{u}_i; \theta, \eta) \prod_{k < i \leq n} c(\mathbf{u}_i; \theta', \eta)},$$

where c is the density of C . The statistic Λ_k is carried out through maximum likelihood method, all the necessary conditions of regularity and efficiency have to be assumed, [Lehmann and Casella \(1998\)](#).

If $L_k(\theta, \eta) = \sum_{1 \leq i \leq k} \log c(\mathbf{u}_i; \theta, \eta)$, and $L_k^*(\theta, \eta) = \sum_{k < i \leq n} \log c(\mathbf{u}_i; \theta, \eta)$, then, the likelihood ratio equation can be written as

$$-2 \log(\Lambda_k) = 2(L_k(\hat{\theta}_k, \hat{\eta}_k) + L_k^*(\hat{\theta}_k^*, \hat{\eta}_k^*) - L_n(\hat{\theta}_n, \hat{\eta}_n)).$$

The hypothesis $\mathcal{H}_0^{(3)}$ is rejected for large values of

$$Z_n = \max_{1 \leq k < n} (-2 \log(\Lambda_k)).$$

Pursuing [Gombay and Horváth \(1996\)](#), the following approximation holds:

$$\mathbb{P}(Z_n^{1/2} \geq x) \approx \frac{x^p \exp(-x^2/2)}{2^{p/2} \Gamma(p/2)} \cdot (HL - \frac{p}{x^2} HL + \frac{4}{x^2} + O(\frac{1}{x^4})),$$

as $x \rightarrow \infty$, where $HL = \log \frac{(1 - g_n)(1 - l_n)}{g_n l_n}$, $g_n = l_n = (\log n)^{3/2}/n$, [Dias and Embrechts \(2004\)](#).

If we assume that there is exactly one change point, then the estimate for the change time is given by $\hat{k}_n = \min\{1 \leq k < n : Z_n = -2 \log(\Lambda_k)\}$.

Considering that the change-point test has less power for small changes, we analyze the dependence more specifically by assuming a time-varying behavior for the corresponding parameter. In order to show how it works, we provide now the dynamics of the parameters for the copulas that we use in the applications. The definitions of the copulas are recalled in an Annex.

Using the dynamic Gaussian copula, we define the dynamic correlation as :

$$\rho_t = h^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 h(\rho_{t-1})), \quad (5)$$

where $(x_{1,t})_t$ and $(x_{2,t})_t$ are the samples, r_0, r_1, s_1 the parameters and $h(\cdot)$ the Fisher's transformation such that $h(\rho) = \log\left(\frac{1+\rho}{1-\rho}\right)$, to ensure that $-1 < \rho < 1$.

If we work with the dynamic Student t -copula, the dynamic degrees of freedom ν can be defined as:

$$\nu_t = l^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 l(\nu_{t-1})), \quad (6)$$

where r_0, r_1, s_1 are parameters and $l(\cdot)$ is a function defined as: $l(\nu) = \log\left(\frac{1}{\nu-2}\right)$.

For the dynamic Gumbel copula, the dynamic parameter δ can be described as:

$$\delta_t = w^{-1}(r_0 + r_1 x_{1,t-1} x_{2,t-1} + s_1 w(\delta_{t-1})), \quad (7)$$

where r_0, r_1, s_1 are parameters and $w(\cdot)$ is a function defined as: $w(\delta) = \log\left(\frac{1}{\nu-1}\right)$.

5 Empirical work

We apply now the above change analysis of dynamic copula to Standard & Poor 500 (S&P500) and Nasdaq indices. The sample data sets contain 2436 daily observations from 4 January, 1993 to 30 August, 2002 for both assets. The log-returns of these two indices are shown in Figure 2.

From Figure 2, it is observed that the outliers of the two underlying log-returns typically occur simultaneously, and almost in the same direction. We observe that both assets fluctuate a lot from the middle of 1997 when the Asian financial crisis burst out.

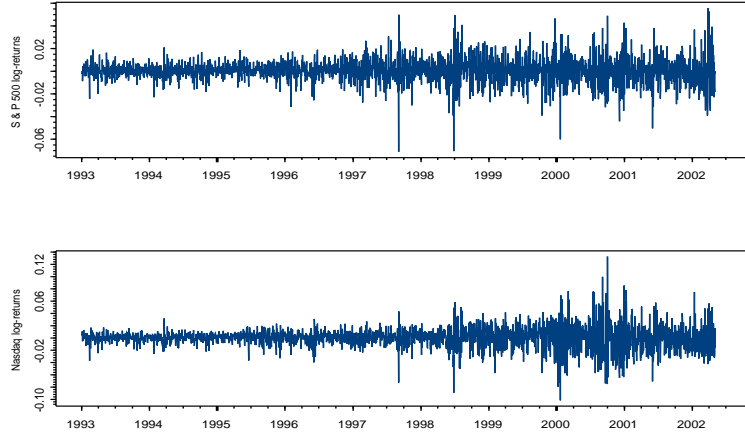


Fig. 2. Log-returns for S&P500 (up) and Nasdaq (down) Indices

Let $r_{i,t}$ ($i = 1, 2$) be the daily log-returns for S&P500 and Nasdaq respectively. In order to filter the observed instability, we fit a univariate GARCH(1,1) model to each log-return series, that is:

$$\begin{aligned}
 r_{i,t} &= \mu_i + \xi_{i,t} \quad \text{with} \quad \xi_{i,t} = \sigma_{i,t} \varepsilon_{i,t}, \\
 \sigma_{i,t}^2 &= \alpha_{i,0} + \alpha_{i,1} \varepsilon_{i,t-1}^2 + \beta_{i,1} \sigma_{i,t-1}^2, \\
 \varepsilon_{i,t} | \varphi_{i,t-1} &\sim N(0, 1),
 \end{aligned} \tag{8}$$

where μ_i is the drift, $\alpha_{i,0}$, $\alpha_{i,1}$, $\beta_{i,1}$ are parameters in \mathbb{R} . The estimation of the parameters using likelihood method are given in Table 1.

Table 1. Estimates of GARCH(1,1) parameters

Parameter	S&P500	Nasdaq
μ	6.013e-04 (1.633e-04)	9.395e-04 (2.116e-04)
α_0	6.018e-07 (1.579e-07)	1.486e-06 (2.877e-07)
α_1	7.947e-02 (6.670e-03)	1.157e-01 (8.902e-03)
β_1	9.201e-01 (6.761e-03)	8.849e-01 (8.596e-03)

Figures in brackets are standard errors

5.1 Dynamic copula for S&P500 and Nasdaq indices

In order to investigate the dependence between these two data sets, we firstly adjust the best copula for the standard residual-pairs $(\varepsilon_{1,t}, \varepsilon_{2,t})$ over the whole period using AIC criterion. The set of copulas includes Gaussian, Student t , Gumbel, Clayton and Frank copulas. The copulas fitting is given in Table 2. Although Student t copula has the smallest AIC value, the estimation is unfortunately not convergent, therefore, Gaussian copula provides the best copula for the whole sample.

Table 2. Copula fitting results

Copula	Parameter	AIC	Convergence
Gaussian	8.116e-01 (2.684e-02)	-2615.196	T
Student t	8.143e-01 (3.384e-02); 13.668 (5.078e-01)	-2642.88	F
Gumbel	2.461 (4.090e-02)	-2505.374	T
Clayton	1.659 (5.280e-02)	-1867.982	T
Frank	8.391 (1.878e-01)	-2419.844	T

Figures in brackets are standard errors, for Student t copula, the first parameter is correlation, the second one is degree of freedom, and “T” means “True”, “F” means “Fault”.

In a first step, we test the stability of this copula. We use the test developed in Section 3.1 and the statistics \mathcal{T} in Equation (3). Here, we assume that the true copula is the Gaussian one specified in Table 2. To apply the test, we choose a kernel function K given by

$$K(\mathbf{u}) = \left(\frac{15}{16}\right)^2 \prod_{i=1}^2 (1 - u_i^2)^2 \mathbf{1}_{\{u_i \in [0,1]\}},$$

a bandwidth $\hat{h}_n = \frac{\sqrt{(\sigma_1^2 + \sigma_2^2)/2}}{n^{1/6}}$, and σ_l^2 will be the empirical variance of \hat{F}_l ($l = 1, 2$). Furthermore, for the points $(\mathbf{u}_j)_{j=1,2,\dots,m}$ in Assumption 4, we choose $m = 81$ points on the uniform grid with the type of $(1/10, 2/10, \dots, 9/10) \times (1/10, 2/10, \dots, 9/10)$.

Using this approach, the p -value for the null hypothesis $\mathcal{H}_0^{(1)}$ is equal to 0. Thus the null hypothesis is rejected and the copula for the data set does not remain static.

In a second step, we detect the changes of copula's family using the binary segmentation procedure described in Section 4.1. Through deciding the best copulas on the subsamples divided by the binary segmentation, all of the change time for the copula's family are detected. The results are given in Table 3.

Table 3. Changes of copula's family

Period	Copula	Parameter	Change time
04/01/93-24/10/97	Gaussian	7.716e-01 (3.632e-02)	-
24/10/97-11/01/99	Student t	8.497e-01 (3.636e-02); 8.355 (2.071)	24 Oct. 1997
11/01/99-18/08/99	Gumbel	3.06275 (2.027e-01)	11 Jan. 1999
18/08/99-06/12/99	Gaussian	8.429e-01 (1.595e-01)	18 Aug. 1999
06/12/99-24/03/00	Student t	6.317e-01 (1.462e-01); 14.564 (1.644)	6 Dec. 1999
24/03/00-09/08/00	Gumbel	2.81704 (2.384e-01)	24 Mar. 2000
09/08/00-22/12/00	Gaussian	8.630e-01 (1.481e-01)	9 Aug. 2000
22/12/00-20/02/01	Student t	9.115e-01 (2.844e-01); 1.693383 (9.324e-01)	22 Dec. 2000
20/02/01-08/06/01	Gaussian	8.673e-001 (1.709e-01)	20 Feb. 2001
08/06/01-30/08/02	Student t	8.948e-01 (1.200e-01); 24.506 (1.134)	8 Jun. 2001

“Period” shows the start and end time of the observations within the corresponding subsamples, in the form of Day/Month/Year, where “Year” is represented by the last two numbers of the year, i.e., “99” represents the year 1999 for instance. Figures in brackets are standard errors, and for Student t copula, the first parameter is correlation, the second one is degree of freedom.

The result in Table 3 provides the change period for copula's family that coincide with some financial incidents:

- 24 Oct. 1997: copula family changes from Gaussian to Student t . This date corresponds to 27 October, 1997 when the Asian financial crisis came to a head.

- 11 Jan. 1999: copula family changes from Student t to Gumbel. This date corresponds to the introduction of Euro as the unit European currency.
- 24 Mar. 2000: copula family changes from Student t to Gumbel. This date corresponds to the technology-heavy Nasdaq stock market peaked on 10 Mar. 2000 and S&P 500 peaked on 24 Mar. 2000.
- 8 Jun. 2001: copula family changes from Gaussian to Student t . This date corresponds to the subsequent 9.11 attacks and the recession lasted from March 2001 to November 2001 in the United States.

Thirdly, for each corresponding period within which the copula’s family does not change, we detect the change points for the copula’s parameters in the way introduced in Section 4.2. We provide the results in Table 4. $z_n^{1/2}$ is the corresponding observation value for the statistics $Z_n^{1/2}$.

Table 4. Change-point for copula’s parameters

Period	Copula	$z_n^{1/2}$	P	$\mathcal{H}_0^{(3)}$	Change time
04/01/93-24/10/97	Gaussian	3.134	7.250e-02	×	15 Oct. 1997
24/10/97-11/01/99	Student t	1.240	4.214e-01	✓	-
11/01/99-18/08/99	Gumbel	2.253	3.471e-01	✓	-
18/08/99-06/12/99	Gaussian	2.938	6.331e-02	×	1 Dec. 1999
06/12/99-24/03/00	Student t	1.255	6.648e-01	✓	-
24/03/00-09/08/00	Gumbel	2.761	1.054e-01	✓	-
09/08/00-22/12/00	Gaussian	2.298	2.829e-01	✓	-
22/12/00-20/02/01	Student t	2.547	1.272e-01	✓	-
20/02/01-08/06/01	Gaussian	3.398	1.702e-02	×	4 Jun. 2001
08/06/01-30/08/02	Student t	1.818	7.743e-01	✓	-

“Period” shows the start and end time of the observations within the corresponding subsamples, in the form of Day/Month/Year, where “Year” is represented by the last two numbers of the year, i.e., “99” represents the year 1999 for instance. P denotes the probability $P(Z_n^{1/2} > z_n^{1/2})$ in Section 4.2, the null hypothesis $\mathcal{H}_0^{(3)}$ is rejected at a 10% level, we simply denote “✓” as “not reject” and “×” as “reject”.

The change points for the copula’s parameter shown in Table 4 reflect some financial events, which can be described as:

- 15 Oct. 1997: corresponds to the Asian financial crisis beginning from July 1997;
- 1 Dec. 1999: corresponds to the preparation of the unit European currency, euro;
- 4 Jun. 2001: corresponds to the recession beginning from March 2000 to November 2001, as the real gross domestic product in the United States dropped by 0.2% total from the fourth quarter of 2000;

Finally, as the above change-point analysis only detects “large” changes in the parameters, we further study the dynamic parameters using the appropriate time-varying functions introduced in Equation (5), (6) and (7). The results are given in Table 5.

5.2 Risk management strategy

Our systematic change analysis for the dynamic copula can be tractably applied to measure the dynamics in the dependence structure of the financial data. Now we compute the simulated VaR and ES measures in a dynamic way. For a given probability level α , $0 < \alpha < 1$, VaR_α is simply the maximum loss that is exceeded over a specified period with a level of confidence $1 - \alpha$. If X is a random return with distribution function F_X , then

$$F_X(\text{VaR}_\alpha) = P\{X \leq \text{VaR}_\alpha\} = \alpha.$$

Thus, losses lower than VaR_α occur with probability α . For the other measure ES (Expected Shortfall), it represents the expectation of loss knowing that a threshold is exceeded, for instance VaR_α , and we define it as:

$$\text{ES}_\alpha(X) = E\{X|X \leq \text{VaR}_\alpha\}.$$

For the portfolio of S&P500 and Nasdaq with equal weight, we compare the VaR and ES values using the static copula and the dynamic copula. For the static copula, we choose the Gaussian copula given in Table 2 corresponding to the whole period. We use the dynamic copula obtained through time-varying parameters (given in table 5) over different subsamples assuming that the copula’s family does not

Table 5. Estimates for time-varying parameters

Period	Copula	Parameter	r_0	r_1	s_1
04/01/93-24/10/97	Gaussian	dynamic ρ	2.620e-02 (4.961e-02)	4.160e-02 (5.347e-02)	9.735e-01 (2.689e-01)
24/10/97-11/01/99	Student t	$\rho = 8.249e-01$; (2.264e-02)	-	-	-
		dynamic ν	8.915e-01 (8.238e-01)	-1.632e-01 (1.389e-01)	3.313e-01 (1.226e-01)
11/01/99-18/08/99	Gumbel	dynamic δ	-1.263 (2.194e-01)	-5.236e-03 (1.129e-01)	-7.700e-01 (3.890e-02)
18/08/99-06/12/99	Gaussian	dynamic ρ	3.266 (1.495)	3.081e-02 (3.636e-03)	-3.557e-01 (3.291e-02)
06/12/99-24/03/00	Student t	$\rho = 5.433e-01$; (2.938e-02)	-	-	-
		dynamic ν	7.228e-01 (5.977e-01)	4.033e-01 (1.428e-01)	-6.784e-01 (2.584e-01)
24/03/00-09/08/00	Gumbel	dynamic δ	-8.134e-01 (5.823e-01)	-3.892e-02 (3.937e-01)	-4.400e-01 (5.333e-02)
09/08/00-22/12/00	Gaussian	dynamic ρ	3.317 (3.055e-02)	1.104e-01 (2.343e-03)	-3.147e-01 (7.314e-01)
22/12/00-20/02/01	Student t	$\rho = 9.387e-01$; (5.236e-01)	-	-	-
		dynamic ν	-1.922 (1.485)	1.276 (1.893)	-5.806e-01 (5.530e-01)
20/02/01-08/06/01	Gaussian	dynamic ρ	4.236e-02 (6.874e-01)	3.808e-02 (5.258e-02)	9.792e-01 (2.180e-01)
08/06/01-30/08/02	Student t	$\rho = 8.747e-01$; (2.711e-02)	-	-	-
		dynamic ν	-5.764e-01 (5.540e-01)	-3.595e-01 (8.683e-01)	-1.025 (3.303e-01)

“Period” shows the start and end time of the observations within the corresponding subsamples, in the form of Day/Month/Year, where “Year” is represented by the last two numbers of the year, i.e., “99” represents the year 1999 for instance. Figures in brackets are standard errors.

change in each subsample (the families of copulas are provided in Table 3). We calculate the VaR and ES values per 20 days in order to clearly observe the dynamics. The results obtained from the static and dynamic copulas are shown in Figure 3 and Figure 4.

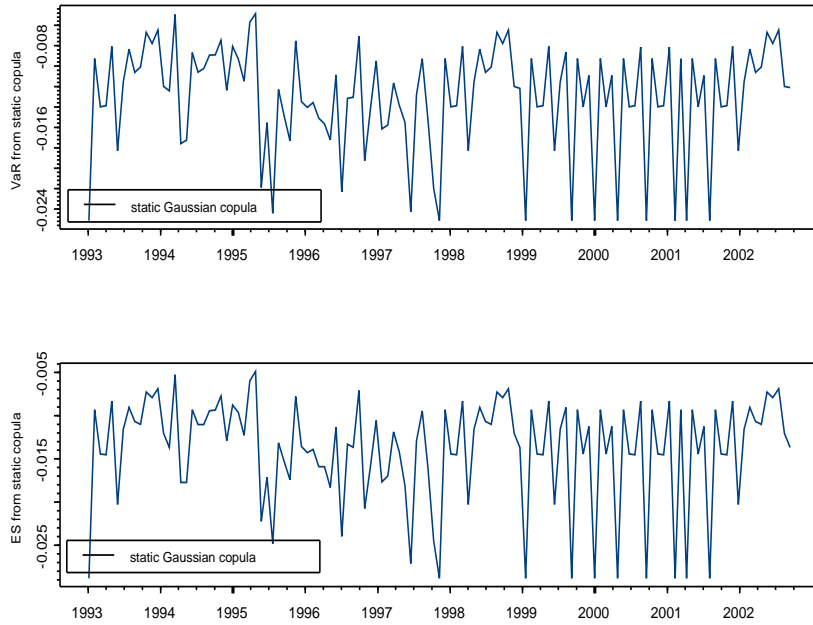


Fig. 3. VaR and ES using static copula for the portfolio of S&P500 and Nasdaq Indices

From Figure 3 and Figure 4, it can be observed that the VaR and ES values fluctuate a lot. Through comparison, some conclusions are summarized below:

1. The dynamics of the VaR and ES using the static copula only come from the volatilities of the GARCH model, while using the dynamic copulas, the dynamics of VaR and ES still depend on the dynamic dependence structure;

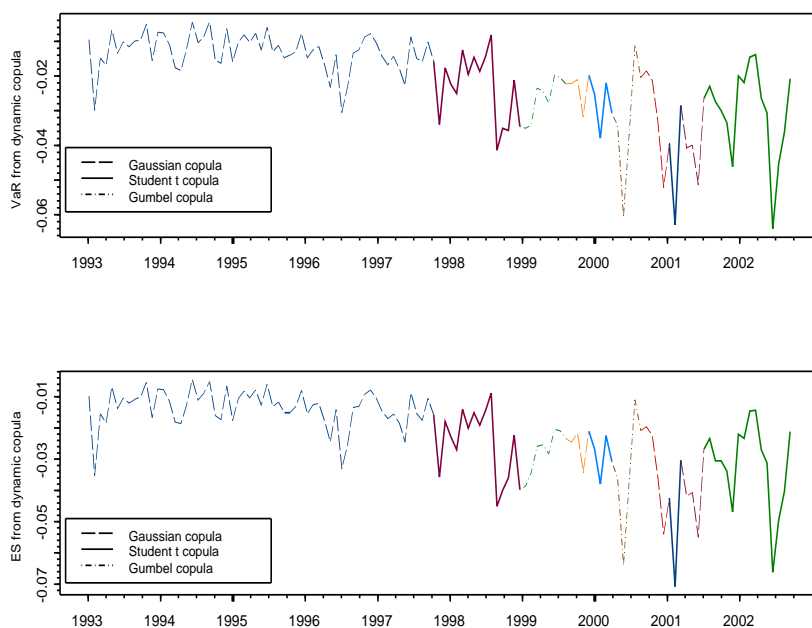


Fig. 4. VaR and ES using dynamic copulas for the portfolio of S&P500 and Nasdaq Indices

2. The VaR and ES from the static copula have generally smaller absolute values than those from the dynamic copulas, which means that the dynamic copula model shows more risk information than the static one. It is very important for portfolio investors who always choose the portfolio with the smallest VaR and ES absolute values. In practice, we observed that it is not appropriate to compute the VaR and ES values using the static copula.
3. After the middle of 1997 when the Asian financial crisis broke out, the VaR and ES values calculated from the dynamic copula vary a lot, while this phenomenon does not distinctly appear when we use the static copula. This means that the dynamic copula model proves better than the static one in terms of the sensitivity to the risk .

From the above remarks, it appears that the dynamic changes inside the dependence structure of a portfolio plays an important role in risk management. Recently we have also observed this fact in multivariate option pricing, using dynamic dependence measured by copulas, [Guégan and Zhang \(2007\)](#).

6 Conclusion

In this paper, we introduce a new approach to detect the best dynamic copula which characterizes the evolution of several data sets. It is based on a series of nested tests concerning the conditional copula and the GOF test. This approach permits to determine the change type of the copula using the binary segmentation procedure, the change-point analysis and the time-varying parameter functions. We illustrate our approach with S&P500 and Nasdaq indices. The empirical result presented the changes of copula's family as well as the changes of parameters. Furthermore, our approach has been applied to give the dynamic risk measures VaR and ES, which plays an important role in risk management.

7 Annex

7.1 Gaussian copula

The copula of the d -variate normal distribution with linear correlation matrix R is

$$C_R^{\text{Ga}}(\mathbf{u}) = \Phi_R^d(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \dots, \Phi^{-1}(u_d)),$$

where Φ_R^d denotes the joint distribution function of the d -variate standard normal distribution function with linear correlation matrix R , and Φ^{-1} denotes the inverse of the distribution function of the univariate standard Gaussian distribution. Copulas of the above form are called Gaussian copulas. In the bivariate case, we denote ρ as the linear correlation coefficient, then the copula's expression can be written as

$$C^{\text{Ga}}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right\} ds dt.$$

The Gaussian copula C^{Ga} with $\rho < 1$ has neither upper tail dependence nor lower tail dependence.

7.2 Student-t copula

If \mathbf{X} has the stochastic representation

$$\mathbf{X} \stackrel{d}{=} \mu + \frac{\sqrt{\nu}}{\sqrt{S}} \mathbf{Z}, \quad (9)$$

where $\stackrel{d}{=}$ represents the equality in distribution or stochastic equality, $\mu \in \mathbb{R}^d$, $S \sim \chi_\nu^2$ and $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$ are independent, then \mathbf{X} has a d -variate t_ν distribution with mean μ (for $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\Sigma$ (for $\nu > 2$). If $\nu \leq 2$ then $\text{Cov}(\mathbf{X})$ is not defined. In this case we just interpret Σ as the shape parameter of the distribution of \mathbf{X} . The copula of \mathbf{X} given by Equation (9) can be written as

$$C_{\nu,R}^t(\mathbf{u}) = t_{\nu,R}^d(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \dots, t_\nu^{-1}(u_d)),$$

where $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$ for $i, j \in \{1, 2, \dots, d\}$, $t_{\nu,R}^d$ denotes the distribution function of $\sqrt{\nu}\mathbf{Y}/\sqrt{S}$, $S \sim \chi_\nu^2$ and $\mathbf{Y} \sim \mathcal{N}_d(\mathbf{0}, R)$ are independent. Here t_ν denotes the margins of $t_{\nu,R}^d$, i.e., the distribution function of $\sqrt{\nu}Y_i/\sqrt{S}$ for $i = 1, 2, \dots, d$. In the bivariate case with the linear correlation coefficient ρ , the copula's expression can be written as

$$C_{\nu,R}^t(u, v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{ 1 + \frac{s^2 - 2\rho st + t^2}{\nu(1-\rho^2)} \right\}^{-(\nu+2)/2} ds dt.$$

Note that $\nu > 2$. And the upper tail dependence and the lower tail dependence for Student t copula have the equal value.

7.3 Gumbel copula

The Gumbel copula is defined as

$$C_{Gu}(u, v; \delta) = \exp\{-[(-\ln u)^\delta + (-\ln v)^\delta]^{1/\delta}\}, \quad \delta \in [1, \infty).$$

It has the properties:

1. $\delta = 1$ implies $C_{Gu}(u, v; 1) = uv$;
2. As $\delta \rightarrow \infty$, $C_{Gu}(u, v; \delta) \rightarrow \min(u, v)$;
3. Gumbel copula has upper tail dependence: $2 - 2^{1/\delta}$;
4. Gumbel copula has no lower tail dependence.

The Gumbel copula belongs to the Archimedean copula, [Joe \(1997\)](#) and [Nelsen \(1999\)](#).

Bibliography

- Akaike, H., 1974. A new look to the statistical model identification. *IEEE Transactions on Automatic Control*, AC-19, 716-723.
- Anderson, T. W., 1969. Statistical inference for covariance matrices with linear structure. *Proceeding in the Second International Symposium on Multivariate Analysis*, ed. P. R. Krishnaiah. Academic Press, New York, 55-66.
- Caillault, C. and Guégan, D., 2007. Forecasting VaR and Expected Shortfall using dynamical systems: A risk management strategy, *Frontiers in Finance*, to appear.
- Caillault, C., Guégan, D., 2005. Empirical estimation of tail dependence using copulas: Application to Asian markets. *Quantitative Finance*, 5, 489-501.
- Csörgő, M., Horváth, L., 1997. *Limit Theorems in Change-point Analysis*. Wiley, Chichester.
- Dias, A., Embrechts, P., 2004. Dynamic copula models for multivariate high-frequency data in finance. Manuscript, ETH Zurich.
- Embrechts, P., McNeil, A., Strausmann, D., 2001. Correlation and dependence in risk management: properties and pitfalls. *Risk Management: Value at Risk and Beyond*, ed. M.A.H. Dempster, Cambridge University Press, 176-223.
- Fermanian, J-D., 2005. Goodness of fit tests for copulas. *Journal of multivariate analysis* 95, 119-152.
- Gombay, E., Horváth, L., 1996. On the rate of approximations for maximum likelihood test in change-point models. *Journal of Multivariate Analysis* 56, 120-152.
- Gombay, E., Horváth, L., 1999. Change-points and bootstrap. *Environmetrics* 10, 725-736.
- Granger, C.W.J., Teräsvirta, T., Patton, A. J., 2006. Common factors in conditional distributions for bivariate time series. *Journal of Econometrics* 132, 43-57.
- Guégan, D., 2007. Global and local stationary modelling in finance: Theory and empirical evidence. Working Paper, CES, Université Paris 1 Panthéon - Sorbonne, France, 2007.54.
- Guégan, D., Zhang, J., 2007. Pricing Bivariate Option under GARCH-GH Model with Dynamic Copula: Application for Chi-

- nese Market, Working Paper, CES, Université Paris 1 Panthéon - Sorbonne, France, 2007.57.
- Joe, J., 1997. *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.
- Jondeau, E., Rockinger, M., 2006. The copula-GARCH model of conditional dependencies: An international stock market application. *Journal of International Money and Finance* 25, 827-853.
- Lehmann, E. L., Casella, G., 1998. *Theory of point estimation*, second edition. Springer, New York.
- Nelsen, R., 1999. An introduction to copulas. *Lecture Notes in Statistics* 139.
- Patton, A. J., 2006. Modelling asymmetric exchange rate dependence. *International Economic Review* 47, 527-556.
- Silverman, B. W., 1986. *Density Estimation for Statistics and Data Analysis*. Chapman & Hall, CRC.
- Sklar, A., Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de L'Université de Paris*. 8, 229-231.
- Vostrikova, L. J., 1981. Detecting "disorder" in multidimensional random processes. *Soviet Mathematics Doklady* 24, 55-59.