UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Documents de Travail du
Centre d'Economie de la Sorbonne

C E S Working Papers

CES
Centre d'Économie de la Sorbonne
UMR 8174

Dropping Rational Expectations

Lionel DE BOISDEFFRE

2021.09

cnrs dépasser les frontières

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.cnrs.fr/
ISSN : 1955-611X

Dropping Rational Expectations

Lionel de Boisdeffre,[1]

(February 2021)

Abstract

We consider a pure-exchange sequential economy, where uncertainty prevails and agents, possibly asymmetrically informed, exchange commodities, on spot markets, and securities of all kinds, on typically incomplete financial markets. Consumers have private characteristics, anticipations and beliefs, and no model to forecast prices. We show that they face an incompressible uncertainty, represented by a so-called 'minimum uncertainty set', which adds to the exogenous uncertainty, on the state of nature, an uncertainty over the price to prevail, on every spot market. Equilibrium is reached when agents expect the 'true' price as a possible outcome on every spot market, and elect optimal strategies, which clear on all markets. We show this sequential equilibrium exists in standard conditions, when agents' anticipations embed the minimum uncertainty set. This outcome is stronger than Radner's (1979), Duffie-Shaffer's (1985) or De Boisdeffre's (2021), which prove the generic existence of equilibrium when agents make perfect forecasts. From an asymptotic argument, our main theorem is derived from De Boisdeffre's (2007), which characterizes the existence of equilibria on purely financial markets by a no-arbitrage condition.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

[1] University of Paris 1 - Panthéon - Sorbonne, 106-112 Boulevard de l'Hôpital, 75013 Paris, France. Email: lionel.de.boisdeffre@wanadoo.fr

1 Introduction

When agents' information is incomplete or asymmetric, the issue of how markets may reveal information is essential and, yet, debated. Quoting Ross Starr (1989), "*the theory with asymmetric information is not well understood at all. In short, the exact mechanism by which prices incorporate information is still a mystery and an attendant theory of volume is simply missing.*" A traditional response is given by the REE (rational expectations equilibrium) model by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the relationship between private information signals and equilibrium prices, along a so-called '*forecast function*'.

In the context of the particular Radner (1979) model of asymmetric information, equilibrium prices are typically distinct across agents' joint information signals. Thus, with a forecast function, agents could theoretically infer all information initially detained by other agents from observing such '*separating*' prices. However, the forecast function is endogeneous, depending on all agents', typically private, characteristics. The eductive process by which agents might infer that forecast function is, therefore, unclear. Moreover, if agents had the '*structural knowledge*' of how equilibrium prices were determined, their using it to correctly infer information at a REE equilibrium would require a reckonning skill, which is seen as unrealistic. The actual process by which markets incorporate information needs, therefore, be studied. Along Cornet-De Boisdeffre (2009) and De Boisdeffre (2016, 2021), markets reveal that information, with no price model, through arbitrage.

Under asymmetric information, the rational expectation assumption states that agents have a price model a la Radner (1979), from which they infer information at a revealing REE. With symmetric information, the rational expectation assumption, also called the '*perfect foresight hypothesis*', states that agents make their trade and consumption plans with the perfect knowledge of future contingent prices, along Radner (1972). On same grounds, this assumption faces similar criticisms as Kurz and Wu's (1996): "*agents need to know the maps from states at future dates to prices in the future and it is entirely unrealistic to assume that agents can find out what this sequence of maps is.*" Radner (1982) himself acknowledges that it "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*". Yet, the classical sequential equilibrium relies on perfect foresight.

The current paper drops both rational expectation assumptions and extends the concept of sequential equilibrium. Agents with no price map learn information from markets, whenever possible, along a rational inference behaviour called the '*no-arbitrage principle*' (see De Boisdeffre, 2016 and 2021). Agents' anticipations of admissible states and prices are exogenous. After being refined from the no-arbitrage principle, such anticipations preclude arbitrage on financial markets. Accordingly, the extended concept of equilibrium only requires that anticipations be self-fulfilling ex post, instead of perfect. Namely, a sequential equilibrium is reached when agents anticipate the true spot price on every spot market as a *possible*, instead of *certain*, outcome and all other conditions of equilibrium hold unchanged. We introduce and refer to such a sequential equilibrium as a '*correct foresight equilibrium (CFE)*'.

At a CFE, agents no longer have a price map to refer to and, yet, face no bankruptcy or unexpected price or event across periods. Therefore, cautious rational agents would typically anticipate a continuum of admissible spot prices, making the model infinite dimensional. This complexity may explain why standard sequential equilibrium models always assumed rational expectations. To our best knowledge, standard models explore no alternative condition than a price map to insure that agents made correct inferences and self-fulfilling anticipations. The current paper attempts to fill this gap. It recalls what information markets may reveal to agents with no price map. It states a condition, which insures that their anticipations are self-fulfilling and that equilibria exist, whatever the financial and information structure. These outcomes of the paper differ from the classical models', namely, the models of rational expectations, bounded rationality and temporary equilibrium.

As a consequence of their having no price map or structural knowledge, we show that agents face an incompressible uncertainty over the set of future spot prices, called the '*minimum uncertainty set*'. This set is non-empty and consists of all possible equilibrium spot prices at agents' arbitrary beliefs, which are private. This price uncertainty is akin to Kurz and Wu's (1994 and 1996), when described as "*primarily endogenous and internally propagated (...) generated by the actions and beliefs of the agents (...) and by their uncertainty about the actions of other agents*".

That minimum uncertainty set, or a bigger set, might be inferred, we argue, by a tradehouse or financial institution from observing and treating past data on long time series. Consumers themselves would not have the required computational capacities. But they could build personal beliefs on public institutions' forecasts. No agent or institution would forecast equilibrium prices with certainty, because this would typically require to know every agent's private characteristics and actions ex ante. Only a set of possible clearing-market prices might be inferred, namely, the minimum uncertainty set, possibly endowed with a probability distribution. The precise location of future prices in that set would remain uncertain (see Section 3).

The correct foresight equilibrium (CFE) is thus defined as De Boisdeffre's (2007) equilibrium, except for agents' expectations, which need no longer be unique in every state, but form sets containing the true spot prices. The CFE, we argue, reconciles into one concept the sequential and temporary equilibria, which Grandmont (1982) describes as dichotomic. It is sequential, since anticipations are self-fulfilling. It is temporary, since forecasts are exogenously given and not endogenous to the model.

Along our main Theorem, whether financial assets be nominal or real and whether agents shared the same beliefs or disagreed about the future, a CFE exists whenever all consumers' anticipations embed the minimum uncertainty set. The current paper proves this theorem in a model, which drops all forms of rational expectations.

The approach to information transmission and equilibrium, proposed hereafter and in our earlier papers, seems to model agents' actual behaviours. Consumers have limited observational and reckonning capacities, hence, no forecast function. They are unaware of the primitives of the economy, make exogenous anticipations and face uncertainty over future spot prices, as on actual markets. They may infer a unique coarsest arbitrage-free refinement of prior expectations from observing trade, along De Boisdeffre (2016). However, once reached, this refinement can no longer be improved and market forces, driven by prices, lead to equilibrium.

The paper is organized as follows: Section 2 presents the model. Section 3 states the existence Theorem and discusses its specific anticipation assumption. Section 4 proves the Theorem. An Appendix proves technical Lemmas.

2 The model

Throughout the paper, we consider a two-period pure exchange economy, where agents face uncertainty, receive information signals and exchange consumption goods and financial assets on markets. The sets, I, J, L and S, respectively, of consumers, securities, goods and states of nature are all given and finite. The first period is referred to as $t = 0$ and the second, as $t = 1$. At $t = 0$, agents are uncertain which state of nature will randomly prevail at $t = 1$. At $t = 1$, all uncertainty is removed.

The non-random state, at $t = 0$, is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$ for every subset, Σ, of S. Perishable goods, $l \in L$, may be exchanged on spot markets for consumption purposes at both dates. Financial assets, also called securities, are exchanged at $t = 0$ and pay off in goods and/or in cash at $t = 1$. We let $l = 0$ be the unit of account (payoff in cash) and $L' := \{0\} \cup L$ be the set of all payoffs at $t = 1$.

2.1 Markets, information and beliefs

At $t = 0$, each agent, $i \in I$, receives a private information signal, $S_i \subset S$, which informs her correctly that no state $s \in S \backslash S_i$ will prevail at $t = 1$. Hence, the pooled information set, henceforth denoted by $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, is non-empty. The information structure, $(S_i) := (S_i)_{i \in I}$, is set as given throughout. Agents are unaware of the primitives of the economy and of other agents' beliefs, information and actions. They fail to know how market prices are determined and, therefore, face uncertainty over future spot prices. At $t = 0$, the generic i^{th} agent elects a private set of anticipations, in each state $s \in S_i$, out of a set of admissible prices, $P := \{p \in \mathbb{R}^L_{++} : \|p\| = 1\}$.

We refer to $\Omega := S \times P$ as the set of forecasts and denote by ω its generic element, and by $\mathcal{B}(\Omega)$ its Borel σ-algebra. A forecast, $\omega := (s, p) \in \Omega$, is thus a pair of a random state, $s \in S$, and a conditional spot price, $p \in P$, expected (as possible) in that state.

Remark 1 Strictly positive prices in P are related to strictly increasing preferences, as assumed below. For simplicity, but w.l.o.g., the price set, P, normalizes all agents' price expectations to one. In each state, this common value of one could be replaced by any other positive value without changing the model's properties.

Agents may transfer wealth across periods and states by exchanging, at $t = 0$, finitely many assets, $j \in J$, which pay off at $t = 1$, conditionally on the realization of the forecast. In each state, financial payoffs may be nominal (i.e., in cash) or real (in goods) or a mix of both. All assets' payoffs define a payoff function, $V : \Omega \to \mathbb{R}^J$, which is henceforth given and continuous, from the definition. Thus, at asset price, $q \in \mathbb{R}^J$, agents may buy or sell unrestrictively portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of conditional payoffs in cash across forecasts, $\omega \in \Omega$. We now define anticipations.

Definition 1 *An anticipation set is a closed subset of* $\Omega := S \times P$. *A collection of anticipation sets,* $\Omega_i := \cup_{s \in S_i} \{s\} \times P^i_s$, *defined for each* $i \in I$, *is an anticipation structure if the following conditions hold:*

(a) $\forall (i, s) \in I \times S_i$, $P^i_s \neq \varnothing$;

(b) $\forall s \in \underline{\mathbf{S}}$, $\cap_{i \in I} P^i_s \neq \varnothing$.

We let $\mathcal{AS}$ *be the set of anticipation structures. A structure,* $(\widetilde{\Omega}_i) \in \mathcal{AS}$, *is called a refinement of* $(\Omega_i) \in \mathcal{AS}$, *and denoted by* $(\widetilde{\Omega}_i) \leq (\Omega_i)$, *if it is smaller than* (Ω_i) *for the inclusion relation. A structure of beliefs is a collection of probabilities,* π_i, *over* $(\Omega, \mathcal{B}(\Omega))$, *called beliefs and defined for each* $i \in I$, *whose supports,* $supp(\pi_i)$, *define an anticipation structure,* $(supp(\pi_i)) \in \mathcal{AS}$. *We let* $\mathcal{SB}$ *be the set of structures of beliefs.*

Remark 2 The support of a belief, π, is defined as the set of forecasts, $\omega \in \Omega$, whose neighbourhoods, U, always satisfy $\pi(U) > 0$. Beliefs have closed supports from the definition. Thus, anticipation sets, seen as supports of beliefs, are necessarily closed. The condition of Definition 1 that they be closed does not restrict the model.

2.2 The agent's behaviour and the concept of equilibrium

The generic i^{th} agent (for $i \in I$) is granted an endowment in form of a vector, $e_i := (e_{is}) \in \mathbb{R}_+^{L \times S_i'}$, promising the commodity bundles, $e_{i0} \in \mathbb{R}_+^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_+^L$, in each state $s \in S_i$, if it prevails. When they elect their strategies, at $t = 0$, agents have reached a structure of beliefs, $(\pi_i) \in \mathcal{SB}$, defining their anticipation structure, $(supp(\pi_i)) \in \mathcal{AS}$. The structures $(\pi_i) \in \mathcal{SB}$ or $(supp(\pi_i)) \in \mathcal{AS}$ are exogenous and assumed (costlessly from De Boisdeffre (2016, 2021)) to be arbitrage-free, that is, to meet the following condition of absence of future arbitrage opportunity (AFAO):

$$\nexists (z_i) \in \mathbb{R}^{J \times I} : \sum_{i \in I} z_i = 0 \, and \, V(\omega_i) \cdot z_i \geqslant 0, \ \forall (i, \omega_i) \in I \times supp(\pi_i), \ \textit{with one strict inequality}.$$

The generic i^{th} agent's consumption set, which depends on her belief, π_i, is denoted by X_{π_i} and defined as the set of continuous maps, $x : \{0\} \cup supp(\pi_i) \to \mathbb{R}_+^L$, which relate state $s = 0$ to the first period consumption, $x_0 \in \mathbb{R}_+^L$, and every forecast, $\omega \in supp(\pi_i)$, to the conditional consumption, $x_\omega \in \mathbb{R}_+^L$ at $t = 1$, chosen if ω obtains.

The asset and commodity prices at $t = 0$ are observed by every agent and denoted by $\omega_0 := (p_0, q)$. As in Remark 1, we restrict w.l.o.g. their admissible values to the set $P_0 := \{p \in \mathbb{R}_+^L : \|p\| \leqslant 1\} \times \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$. Given her belief, π_i, and the observed prices, $\omega_0 := (p_0, q) \in P_0$, at $t = 0$, the generic i^{th} agent's budget set is $B_{\pi_i}(\omega_0) :=$ $\{ \ (x, z) \in X_{\pi_i} \times \mathbb{R}^J : \ p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z, \quad p_s \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z, \ \forall \omega := (s, p_s) \in supp(\pi_i) \ \}$.

The generic agent, $i \in I$, has ordered preferences, represented, ex post, by a utility function, $u_i : \mathbb{R}_+^2 \to \mathbb{R}_+$, over her consumptions at both periods, and, ex ante, by the V.N.M. utility function: $x \in X_{\pi_i} \mapsto U_i^{\pi_i}(x) := \int_{\omega \in \Omega} u_i(x_0, x_\omega) d\pi_i(\omega)$.

The above economy, denoted by $\mathcal{E}_{(\pi_i)} = \{(I, S, L, J), V, (S_i), (\pi_i), (e_i), (u_i)\}$, retains the small consumer price-taker hypothesis, by which no single agent may, alone, have a significant impact on prices. It is standard if it meets the following Assumptions:

- ***A1*** *(strong survival):* $\forall i \in I,\ e_i \in \mathbb{R}_{++}^{L \times S'_i}$;
- ***A2*** *for each* $i \in I$, u_i *is continuous, strictly concave and strictly increasing, namely:* $[(x, y, x', y') \in \mathbb{R}_+^{4L},\ (x, y) \leqslant (x', y'),\ (x, y) \neq (x', y')] \Rightarrow [u_i(x', y') > u_i(x, y)]$.

Remark 3 Strict concavity is retained in Assumption *A2* to alleviate the proof of a correspondence selection amongst optimal strategies (see proof of Lemma 4).

In the economy $\mathcal{E}_{(\pi_i)}$, each consumer elects an optimal strategy in her budget set. This leads to the following concept of equilibrium:

Definition 2 *Given* $(\pi_i) \in \mathcal{SB}$, *a collection of prices,* $\omega_0 := (p_0, q) \in P_0$ *and* $p := (p_s) \in \mathcal{P} := P^{\underline{\mathbf{S}}}$, *and strategies,* $(x_i, z_i) \in B_{\pi_i}(\omega_0)$, *defined for each* $i \in I$, *is a sequential equilibrium of the economy,* $\mathcal{E}_{(\pi_i)}$, *or correct foresight equilibrium (CFE), if:*

(a) $\forall i \in I,\ (x_i, z_i) \in \arg\max_{(x,z) \in B_{\pi_i}(\omega_0)}\ U_i^{\pi_i}(x)$;

(b) $\forall (i, s) \in I \times \underline{\mathbf{S}})$, $(s, p_s) \in supp(\pi_i)$;

(c) $\forall s \in \underline{\mathbf{S}}'$, $\sum_{i \in I} (x_{i\omega_s} - e_{is}) = 0$, *where* $\omega_s := (s, p_s)$;

(d) $\sum_{i \in I} z_i = 0$.

3 The existence Theorem

This Section introduces a set of incompressible uncertainty and states the paper's main Theorem. A discussion of its specific anticipation assumption follows.

3.1 Endogenous uncertainty and the existence of equilibrium

Agents' incompressible uncertainty, sketched above, is characterized by a so-called and never empty '*minimum uncertainty set*', defined hereafter. Theorem 1, below, shows that equilibrium always exists, in a standard economy, when agents' anticipations embed that set of minimum uncertainty. This full existence result differs from the generic ones of the standard models, based on rational expectations.

Definition 3 *Let* Λ *be the set of prices,* $p := (p_s) \in \mathcal{P} := P^{\underline{\mathbf{S}}}$*, which support a CFE, that is, are equilibrium prices of a standard economy,* $\mathcal{E}_{(\widetilde{\pi}_i)}$*, for some beliefs,* $(\widetilde{\pi}_i) \in \mathcal{SB}$*. The (possibly empty) set of forecasts,* $\Delta := \{\omega \in \Omega : \exists p := (p_s) \in \Lambda,\ \exists s \in \underline{\mathbf{S}},\ \omega = (s, p_s)\}$*, which support a CFE, is called the minimum uncertainty set.*

The set Δ is that of forecasts, which may be shared by agents and clear markets tomorrow, for at least one structure of beliefs today. The following Theorem 1 shows that this set is never empty. Lemma 1 and Assumption *A3*, stated before, rely on the convention that the empty set is included in all other sets.

Lemma 1 *Under Assumptions A1-A2, there exists* $\delta \in \mathbb{R}_{++}$ *such that:* $\Delta \subset \underline{\mathbf{S}} \times [\delta, 1]^L$*.*

Proof See the Appendix. □

Assumption A3 *(correct foresight): beliefs,* $(\pi_i) \in \mathcal{SB}$*, satisfy* $\Delta \subset \cap_{i \in I}\ supp(\pi_i)$*.*

Theorem 1 *Let* $(\pi_i) \in \mathcal{SB}$ *be given. Under Assumptions A1-A2-A3, the economy,* $\mathcal{E}_{(\pi_i)}$*, admits a CFE. The set* Δ *is, therefore, non-empty.*

Proof See Section 4, below. □

3.2 Endogenous uncertainty and how to reach correct anticipations

Along Theorem 1, as long as agents have correct foresight, a CFE exists whatever their beliefs. That is, markets always clear ex post at some common forecast. We argue why Δ is a set of '*minimum uncertainty*' and how it could be assessed.

On the first issue, when beliefs are private, possibly changing, no prospective clearing-market price should be ruled out *a priori* by cautious rational agents. This outcome would remain if agents with no price map were symmetrically informed and knew the structural characteristics of the economy. Any common clearing-market forecast might obtain tomorrow, in relation to some unknown collection of beliefs today. From a theoretical viewpoint, Δ is, indeed, a set of minimum uncertainty.

The second issue questions the possibility that agents inferred the set Δ, or a bigger set. This problem might be solved empirically, from treating past data.

The above model specifies *normalized* spot prices, in the sense of Remark 1. It is often possible to observe past prices and reckon their *relative* (normalized) values, at many dates and in a wide array of situations, which typlically replicate over time, like the realizations of random states. If time series are long enough, it seems sensible to assume, in each type of situations (or '*state* s'), that the true spot price in state s (up to a scale factor) belongs to the convex hull of its empirical values. Under this assumption, an embedding of the set Δ may easily be constructed.

To be more specific, assume that long series of prices between two dates, $t_1 \in \mathbb{N}$ and $t_2 >> t_1$, are observed by a so-called '*tradehouse*', and treated as follows. The tradehouse partitions the events occuring between t_1 and t_2, into a finite set, $\underline{\mathbf{S}}$, of realizable states. Thus, at each date, $t \in \{t_1, t_1 + 1, ..., t_2\}$, the tradehouse knows which state, $s \in \underline{\mathbf{S}}$, prevails. For each $s \in \underline{\mathbf{S}}$, it observes the number of times, N_s, that state s occurs between t_1 and t_2. For each $s \in \underline{\mathbf{S}}$, it deduces the time series, $\{p_s^k\}_{1 \leqslant k \leqslant N_s} \subset P$, of the empirical spot price in state s, its convex hull, C_s, and the set $P_s := \{p \in P : \exists p_s \in C_s, \ p = \frac{p_s}{\|p_s\|}\}$. The above assumption states that $\Delta \subset \cup_{s \in \underline{\mathbf{S}}} \{s\} \times P_s$, which seems realistic: the longer the time series, the larger the array of events and beliefs is likely to be encountered, up to spanning all equilibrium possibilities.

A statistical method of this kind and its iterative verification over long series require no price map and need not be performed by consumers themselves. They could be implemented by a financial or public institution, which have greater computational facilities and could also derive useful applications of the method.

On consumer side, agents with no price model should seek to refine their information and take advice from specialists to reduce their uncertainty. Yet, disagreements may persist, with agents receiving typically incomplete or asymmetric (yet correct) pieces of information. As a result, prior anticipations need not be consistent with equilibrium. If not, agents observing markets would update their beliefs from the no-arbitrage principle, up to a unique arbitrage-free refinement. From De Boisdeffre (2016), this refinement never eliminates forecasts, which agents share ex ante. Hence, typical agents would keep idiosyncratic beliefs and share enough uncertainty, represented by a set of common forecasts, to reach equilibrium along Theorem 1.

4 The existence proof

Hereafter, we set as given beliefs, $(\pi_i) \in \mathcal{SB}$, which meet Assumption *A3* and the AFAO Condition. We define a standard economy, $\mathcal{E}_{(\pi_i)} = \{(I, S, L, J), V, (S_i), (\pi_i), (e_i), (u_i)\}$, whose anticipation structure will be denoted by $(\Omega_i) := (supp(\pi_i))$ throughout. The proof proceeds in three steps. Sub-Section 4.1 defines, via finite partitions, a non-decreasing sequence, $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$, of finite refinements of (Ω_i), whose limit is dense in (Ω_i). Sub-Section 4.2 constructs a sequence of finite auxiliary economies, which all admit an equilibrium along De Boisdeffre (2007). Sub-Section 4.3 derives a CFE of the economy $\mathcal{E}_{(\pi_i)}$ from that sequence of auxiliary equilibria.

4.1 Finite partitions of agents' anticipation sets

For every $n \in \mathbb{N}$ (starting from $n = 1$), let $K_n := \mathbb{N}^L \cap [1, 2^{n-1}]^L$. The collection of sets $\Omega_{(n,s,k)} := \{s\} \times (\times_{l \in L}]\frac{k^l-1}{2^{n-1}}, \frac{k^l}{2^{n-1}}])$, defined for every triple $(n, s, k) \in \mathbb{N} \times S \times K_n$, is a partition of $S \times]0, 1]^L$. Let $K_{(i,n)} := \{t := (s, k) \in S_i \times K_n : \Omega^t_{(i,n)} := \Omega_i \, \cap \, \Omega_{(n,s,k)} \neq \varnothing\}$, for every $(i, n) \in I \times \mathbb{N}$. Then, $\{ \, \Omega^t_{(i,n)} \, \}_{t \in K_{(i,n)}}$ is a partition of Ω_i, for every $(i, n) \in I \times \mathbb{N}$, and we define the following sets and maps, which meet the properties of Lemma 2:

- For every triple, $(i, n, t) \in I \times \mathbb{N} \times K_{(i,n)}$, we set as given (exactly) one element, $\omega^t_{(i,n)} \in \Omega^t_{(i,n)} := \Omega_i \, \cap \, \Omega_{(n,s,k)}$, and let $\Omega^n_i := \{\omega^t_{(i,n)}\}_{t \in K_{(i,n)}}$ be a finite subset of Ω_i.
- We define the maps, $\pi^n_i : \Omega^n_i \to \mathbb{R}_{++}$ and $\Phi^n_i : \Omega_i \to \Omega^n_i$, by $\pi^n_i(\omega^t_{(i,n)}) := \pi_i(\Omega^t_{(i,n)})$ and $\Phi^n_i(\omega) := \omega^t_{(i,n)}$, for every $(i, n, t, \omega) \in I \times \mathbb{N} \times K_{(i,n)} \times \Omega^t_{(i,n)}$

Lemma 2 *The following Assertions hold:*

(i) *for every* $(i, n) \in I \times \mathbb{N}$*, the partition* $\{\Omega^t_{(i,n+1)}\}_{t \in K^{n+1}_i}$ *refines* $\{\Omega^t_{(i,n)}\}_{t \in K^n_i}$*; therefore, we may assume that* $\Omega^n_i \subset \Omega^{n+1}_i$*;*

(ii) *for each* $i \in I$*, the set* $\cup_{n \in \mathbb{N}} \Omega^n_i$ *is dense in* Ω_i*;*

(iii) *for every* $i \in I$ *and every* $\omega \in \Omega_i$*,* $\Phi^n_i(\omega)$ *converges uniformly to* ω*;*

(iv) $\exists N \in \mathbb{N} : \forall n \geqslant N$*,* (Ω^n_i) *&* $(\underline{\mathbf{S}} \cup \Omega^n_i)$ *are arbitrage-free along the AFAO Condition.*

Proof Assertions (i) to (iii) are immediate from the definitions. □

Assertion (iv) For each $i \in I$, let $Z_i := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in \Omega_i\}$ and $Z_i^\perp$ be its orthogonal complement. We define $Z_o := \sum_{i \in I} Z_i$ and $Z := \{(z_i) \in \times_{i \in I} Z_i^\perp : \|(z_i)\| = 1\}$. Assume that Assertion (iv) fails. Then, from the definitions and above, it holds that:

$$\forall N \in \mathbb{N} : \exists \; n \geqslant N, \; \exists (z^n_i) \in Z : \textstyle\sum_{i \in I} z^n_i \in Z_o \; \textit{and} \; V(\omega_i) \cdot z^n_i \geqslant 0, \; \forall (i, \omega_i) \in I \times \Omega^n_i.$$

Since Z is compact, the sequence $\{(z^n_i)\}$ may be assumed to converge to $(z^*_i) \in Z$. From Lemma 2-(i)-(ii) and the continuity of V, the above relations yield (for $N \to \infty$):

$\sum_{i\in I} z_i^* \in Z_o$ *and* $V(\omega_i)\cdot z_i^* \geqslant 0,\ \forall (i,\omega_i) \in I\times\Omega_i$.

Since (Ω_i) is arbitrage-free, the above relation, $(z_i^*) \in \times_{i\in I} Z_i^{\perp}$, and the latter yield $(z_i^*) \in \times_{i\in I} Z_i \cap Z_i^{\perp} = \{0\}$ and contradict the above relation $(z_i^*) \in Z$. This contradiction proves that (Ω_i^n), hence $(\underline{\mathbf{S}} \cup \Omega_i^n)$, are arbitrage-free, if $n \in \mathbb{N}$ is large enough. □

4.2 The auxiliary economies, $\mathcal{E}^n$

Given $n \in \mathbb{N}$, we define a finite economy, $\mathcal{E}^n = \{(I, S^n, L, J), V^n, (S_i^n), (e_i^n), (u_i^n)\}$, akin to Cornet-De Boisdeffre's (2002), with two periods, and with the same sets of agents, goods and assets as in Section 2. The information structure is defined as follows:

- $S_i^n := \underline{\mathbf{S}} \cup \Omega_i^n$ is the generic i^{th} agent's information set. Formally, $\underline{\mathbf{S}}$ is defined as the set of realizable states, and Ω_i^n (for each $i \in I$) as the i^{th} agent's idiosyncratic set of unrealizable states.[2] We let $S_i'^n := \underline{\mathbf{S}}' \cup \Omega_i^n$, for each $i \in I$, and $S^n := \cup_{i\in I} S_i^n$.
- In each realizable state, $s \in \underline{\mathbf{S}}$, the generic i^{th} agent has perfect foresight.
- In each formal state, $(s,p) \in \Omega_i^n$, the generic i^{th} agent solely expects $p \in P$.

In the economy $\mathcal{E}^n$ (for $n \in \mathbb{N}$), admissible prices are restricted to the sets $P_0 := \{p \in \mathbb{R}_+^L : \|p\| \leqslant 1\} \times \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$, at $t = 0$, and $\mathcal{P} := P^{\underline{\mathbf{S}}} := \{p \in \mathbb{R}_{++}^L : \|p\| = 1\}^{\underline{\mathbf{S}}}$, at $t = 1$. For all prices, $\omega_0 := (p_0, q) \in P_0$ and $p := (p_s) \in \mathcal{P}$, the generic i^{th} agent's consumption set, budget set, and V.N.M. utility function are, respectively:

$X_i^n := \mathbb{R}_+^{L\times S_i'^n}$, whose generic element is denoted by $x := [(x_s)_{s\in\underline{\mathbf{S}}'}, (x_\omega)_{\omega\in\Omega_i^n}]$;

$$B_i^n(\omega_0, p) := \{\ (x,z) \in X_i^n \times \mathbb{R}^J :\ p_0 \cdot (x_0 - e_{i0}) \leqslant -q\cdot z,\quad p_s \cdot (x_s - e_{is}) \leqslant V(s,p_s)\cdot z,\ \forall s \in \underline{\mathbf{S}}$$
$$\textit{and}\ \ p_s^i \cdot (x_\omega - e_{is}) \leqslant V(\omega)\cdot z,\ \forall \omega := (s, p_s^i) \in \Omega_i^n\ \};$$

[2] We assume costlessly and implicitly the relation $\Omega_i^n \cap \Omega_j^n = \varnothing$, for every $(i,j) \in I\times I\backslash\{i\}$. If it fails, the formal state sets Ω_i^n are replaced at no cost by $\{i\} \times \Omega_i^n$, for each $i \in I$.

and $x \in X_i^n \mapsto\; u_i^n(x) := \frac{1}{n\#\underline{\mathbf{S}}} \sum_{s\in\underline{\mathbf{S}}} u_i(x_0, x_s) \;+ (1-\frac{1}{n}) \sum_{\omega\in\Omega_i^n} u_i(x_0, x_\omega)\pi_i^n(\omega)$.

The concept of equilibrium in the economy $\mathcal{E}^n$ is defined as follows:

Definition 4 *A collection of prices, $\omega_0 := (p_0, q) \in P_0$ and $p := (p_s) \in \mathcal{P}$, and decisions, $(x_i, z_i) \in B_i^n(\omega_0, p)$, for each $i \in I$, is an equilibrium of the economy, $\mathcal{E}^n$, if the following Conditions hold:*

(a) $\forall i \in I,\; (x_i, z_i) \in \arg\max_{(x,z)\in B_i^n(\omega_0,p)}\; u_i^n(x)$;

(b) $\sum_{i\in I}\,(x_{is}-e_{is}) = 0,\; \forall s \in \underline{\mathbf{S}}'$;

(c) $\sum_{i\in I}\; z_i = 0$.

From De Boisdeffre (2007), for every $n \geqslant N$ along Lemma 2, the economy $\mathcal{E}^n$ admits an equilibrium, say $\mathcal{C}^n := \{\omega_0^n := (p_0^n, q^n), p^n, [(x_i^n, z_i^n)]\}$, henceforth set as given, such that $\|\omega_0^n\| \in [1,2]$. The sequence $\{\mathcal{C}^n\} := \{\mathcal{C}^n\}_{n\geqslant N}$ satisfies the following Lemmas:

Lemma 3 *The following Assertions hold:*

(i) the price sequences, $\{\omega_0^n := (p_0^n, q^n)\}$ and $\{p_s^n\}$, may be assumed to converge, say to $\omega_0^* := (p_0^*, q^*) \in P_0$ and $p^* := (p_s^*) \in \mathcal{P} := P^{\underline{\mathbf{S}}}$, s.t. $\omega_s^* := (s, p_s^*) \in \overline{\Delta} \subset \cap_{i\in I}\Omega_i$, for all $s \in \underline{\mathbf{S}}$;

(ii) the sequences $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ and $\{z_i^n\}$, for each $i \in I$, may be assumed to converge, say to $(x_{is}^*)_{s\in\underline{\mathbf{S}}'} \in \mathbb{R}_+^{L\times\underline{\mathbf{S}}'}$ and $z_i^* \in \mathbb{R}^J$, such that $\sum_{i\in I}\,(x_{is}^*-e_{is})_{s\in\underline{\mathbf{S}}'} = 0$, and $\sum_{i\in I} z_i^* = 0$.

Lemma 4 *For all $(i, \omega := (s,p), z) \in I\times\Omega_i\times\mathbb{R}^J$, let $B_i(\omega, z) = \{x \in \mathbb{R}_+^L : p\cdot(x\text{-}e_{is}) \leqslant V(\omega)\cdot z\}$. Using the notations of Lemma 3, the following Assertions hold, for each $i \in I$:*

(i) $\cap_{\omega\in\Omega_i} B_i(\omega, z_i^*) \neq \varnothing$;

(ii) the correspondence $\omega \in \Omega_i \mapsto \arg\max\; u_i(x_{i0}^*, x)$, for $x \in B_i(\omega, z_i^*)$, is a continuous map, whose embedding, $x_i^* : \{0\} \cup \Omega_i \mapsto x_{i\omega}^*$, is a consumption plan, i.e., $x_i^* \in X_{\pi_i}$;

(iii) $x_{i\omega_s^*}^* = x_{is}^*$, *for each* $s \in \underline{\mathbf{S}}$;

(iv) $U_i^{\pi_i}(x_i^*) = \lim_{n\to\infty} u_i^n(x_i^n)$.

Proofs of the Lemmas See the Appendix. □

4.3 An equilibrium of the initial economy

We now prove Theorem 1, via the following Claim 1.

Claim 1 *The collection of prices,* $\omega_0^* \in P_0$ *and* $p^* \in \mathcal{P}$*, and strategies,* (x_i^*, z_i^*)*, for each* $i \in I$*, defined from the above Lemmas 3 and 4, is a CFE of the economy* $\mathcal{E}_{(\pi_i)}$*.*

Proof Let the collection $\mathcal{C}^* := \{\omega_0^*, p^*, [(x_i^*, z_i^*)]\}$ be defined as in Claim 1. From the above Lemmas, $\mathcal{C}^*$ meets Conditions (b)-(c)-(d) of Definition 2 of equilibrium.

We show that $\mathcal{C}^*$ also meets Condition (a) of Definition 2.

From Definition 4, the relations $p_0^n{\cdot}(x_{i0}^n - e_{i0}) \leqslant -q^n{\cdot}z_i^n$ hold, for each $i \in I$ and each $n \geqslant N$, and imply in the limit, from Lemma 3: $p_0^*{\cdot}(x_{i0}^* - e_{i0}) \leqslant -q^*{\cdot}z_i^*$ for each $i \in I$. From Lemma 4, the relations $x_i^* \in X_{\pi_i}$ and $p_s{\cdot}(x_{i\omega}^* - e_{is}) \leqslant V(\omega){\cdot}z_i^*$ hold for each $i \in I$ and every $\omega = (s, p_s) \in \Omega_i$, and imply, from above: $(x_i^*, z_i^*) \in B_{\pi_i}(\omega_0^*)$ for each $i \in I$.

For every $i \in I$, every consumption plan, $x \in X_{\pi_i}$, and every $n \geqslant N$, we let $x^n = [(x_s^n)_{s \in \underline{\mathbf{S}}'}, (x_\omega^n)_{\omega \in \Omega_i^n}] \in X_i^n$ be defined by $x_\omega^n := x_\omega$, for every $\omega \in \Omega_i^n$, $x_0^n := x_0$ and $x_s^n := x_{\omega_s^*}$, for every $s \in \underline{\mathbf{S}}$, where $\omega_s^* := (s, p_s^*) \in \Omega_i$. From the above definitions and Lemmas and the uniform continuity of u_i and x on compact sets, there exists $N_{(\rho,x)} \in \mathbb{N}$, for all $\rho > 0$ and all bounded-valued consumption plan, $x \in X_{\pi_i}$, such that:

$$(I) \qquad |U_i^{\pi_i}(x) - u_i^n(x^n)| < \int_{\omega \in \Omega_i} |u_i(x_0, x_\omega) - u_i(x_0, x_{\Phi_i^n(\omega)})| d\pi_i(\omega) + \rho < 2\rho$$

holds, for every $i \in I$ and every $n \geqslant N_{(\rho,x)}$. Assume, by contraposition, that $\mathcal{C}^*$ fails to meet Condition (a) of Definition 2, that is, there exist $i \in I$, $\varepsilon \in \mathbb{R}_{++}$ and $(X, Z) \in B_{\pi_i}(\omega_0^*)$, such that:

$(II) \qquad U_i^{\pi_i}(x_i^*) + 7\varepsilon < U_i^{\pi_i}(X).$

From Lemma 3 and the fact that Ω_i is closed, X and x_i^* are bounded-valued. Then, from the above relations, the following ones hold, for every $n \geqslant N_{(\varepsilon, x_i^*)} + N_{(\varepsilon, X)}$:

$$(III) \qquad u_i^n(x_i^{*n}) + 3\varepsilon < U_i^{\pi_i}(x_i^*) + 5\varepsilon < U_i^{\pi_i}(X) - 2\varepsilon < u_i^n(X^n) < U_i^{\pi_i}(X) + 2\varepsilon.$$

From Assumption *A1*, the definitions of Ω_i (a closed set) and of the price sets and from the uniform continuity of u_i on a compact set, we may assume that there exists $\gamma > 0$, small enough, such that the above strategy, $(X, Z) \in B_{\pi_i}(\omega_0^*)$, satisfies:

$$p_0^* \cdot (X_0 - e_{i0}) \leqslant -q^* \cdot Z - \gamma \text{ and } p_s \cdot (X_\omega - e_{is}) \leqslant V(\omega) \cdot Z - \gamma, \ \forall \omega := (s, p_s) \in \Omega_i.$$

From Lemmas 2 to 4, and the latter constraints, the relations $(X^n, Z) \in B_i^n(\omega_0^n, p^n)$ and $|U_i^{\pi_i}(x_i^*) - u_i^n(x_i^n)| < \varepsilon$ hold, for $n \in \mathbb{N}$ big enough, say $n \geqslant N' \geqslant N_{(\varepsilon, x_i^*)} + N_{(\varepsilon, X)}$. From Definition 4, the latter relations, joined to relations (I) to (III), yield the following:

$$(IV) \qquad u_i^n(x_i^{*n}) + 3\varepsilon < U_i^{\pi_i}(x_i^*) + 5\varepsilon < u_i^n(X^n) \leqslant u_i^n(x_i^n) < U_i^{\pi_i}(x_i^*) + \varepsilon < u_i^n(x_i^{*n}) + 3\varepsilon,$$

for every $n \geqslant N'$. Relations (IV) embed a contradiction. This contradiction proves that $\mathcal{C}^*$ also meets Condition (a) of Definition 2, hence, from above, is a CFE. $\square$

Appendix

Lemma 1 *Under Assumptions A1-A2, there exists* $\delta \in \mathbb{R}_{++}$ *such that:* $\Delta \subset \underline{\mathbf{S}} \times [\delta, 1]^L$.

Proof We introduce new notations and let, for every $(i, s, x := (x_0, x_s)) \in I \times \underline{\mathbf{S}} \times \mathbb{R}_+^{L \times L}$:

- $y \succ_s^i x$ denote a vector, $y \in \mathbb{R}_+^L$, such that $u_i(x_0, y) > u_i(x_0, x_s)$;
- $\mathcal{A}_s := \{(x_i) := ((x_{i0}, x_{is})) \in \mathbb{R}_+^{L \times L \times I} : \sum_{i \in I} x_i = \sum_{i \in I} (e_{i0}, e_{is})\}$;

- $P_s := \{p \in \overline{P} : \exists j \in I,\ \exists (x_i) \in \mathcal{A}_s,\ such\ that\ (y \succ_s^j x_j) \Rightarrow (p \cdot y \geqslant p \cdot x_{js} \geqslant p \cdot e_{js})\}.$

The proof of Lemma 1 relies on the following Lemmata:

Lemmata 1 *The following Assertions hold:*

(i) $\forall s \in \underline{\mathbf{S}}$*,* P_s *is a closed, hence, compact set;*

(ii) $\exists\ \delta > 0 : \forall s \in \underline{\mathbf{S}}$*,* $P_s \subset [\delta, 1]^L$.

Proof of Lemmata 1 Notice from definitions that $p_s^n \in P_s$ holds for all $n \geqslant N,\ s \in \underline{\mathbf{S}}$.

Assertion (i) Let $s \in \underline{\mathbf{S}}$ and a converging sequence $\{\widetilde{p}^k\}_{k \in \mathbb{N}}$ of elements of P_s be given. Its limit, p, is in the closure, $\overline{P}$, of P. We may assume there exist (a same) $j \in I$ and a sequence, $\{x^k\}_{k \in \mathbb{N}} := \{(x_i^k)\}_{k \in \mathbb{N}}$, of elements of $\mathcal{A}_s$, which meet the condition of the definition of P_s, and which converges to some $x := (x_i) \in \mathcal{A}_s$, a compact set.

The relations $\widetilde{p}^k \cdot (x_{js}^k - e_{js}) \geqslant 0$ hold from the definition, for each $k \in \mathbb{N}$ and yield, in the limit, $p \cdot (x_{js} - e_{js}) \geqslant 0$. We show that (p, j, x) meets the conditions of the definition of P_s (hence, $p = \lim \widetilde{p}^k \in P_s$ and P_s is closed). By contraposition, assume there exists $y \in \mathbb{R}_+^L$, such that $u_j(x_{j0}, y) > u_j(x_{j0}, x_{js})$ and $p \cdot (y - x_{js}) < 0$. Then, we show:

$$(I) \qquad \forall K \in \mathbb{N},\ \exists k > K,\ u_j(x_{j0}^k, y) > u_j(x_{j0}^k, x_{js}^k).$$

If not, one has $u_j(x_{j0}^k, y) \leqslant u_j(x_{j0}^k, x_{js}^k)$, for k big enough, which implies, in the limit $(k \to \infty)$, $u_j(x_{j0}, y) \leqslant u_j(x_{j0}, x_{js})$, in contradiction with the above opposite relation. Hence, relations (I) hold. From the definition of the sequences $\{x^k\}$ and $\{\widetilde{p}^k\}$, relations (I) imply: $\widetilde{p}^k \cdot (y - x_{js}^k) \geqslant 0$, whenever $u_j(x_{j0}^k, y) > u_j(x_{j0}^k, x_{js}^k)$. Hence, in the limit $(k \to \infty)$, $p \cdot (y - x_{js}) \geqslant 0$, in contradiction with the above opposite inequality. This contradiction proves that $p := \lim p^k \in P_s$, hence, P_s is a compact set, for each $s \in \underline{\mathbf{S}}$.□

Assertion (ii) Let $(s, l^*) \in \underline{\mathbf{S}} \times L$ and $p := (p^l) \in P_s$ be given. Let $e := (e^l) \in \mathbb{R}^L$ be such that $e^{l^*} = 1$ and $e^l = 0$ for every $l \in L \backslash \{l^*\}$. We prove that $p^{l^*} = p \cdot e > 0$.

Indeed, let $(p, j, (x_i)) \in P_s \times I \times \mathcal{A}_s$ meet the conditions of the definition of P_s. For every $n > 1$, we let $y^n \in \mathbb{R}^L_+$ be such that $y^n := (1 - \frac{1}{n})x_{js}$. It satisfies $p \cdot (y^n - x_{js}) < 0$ (since $p \cdot x_{js} \geqslant p \cdot e_{js} > 0$, from Assumption *A1* and the definition of P_s).

From Assumption *A2*, there exists $n \in \mathbb{N}$, such that $y := y^n + (1 - \frac{1}{n})e = (1 - \frac{1}{n})(x_{js} + e)$ satisfies $u_j(x_{j0}, y) > u_j(x_{j0}, x_{js})$, which implies: $p \cdot x_{js} \leqslant p \cdot y < p \cdot x_{js} + (1 - \frac{1}{n})p \cdot e$, hence, $p^{l^*} = p \cdot e > 0$. The continuous mapping $\varphi_{(s,l)} : p \in P_s \mapsto p \cdot e \in \mathbb{R}_{++}$ attains a minimum on the compact set P_s, say $\delta_{(s,l)} > 0$, and Assertion (ii) holds for $\delta := \min_{(s,l) \in \underline{\mathbf{S}} \times L} \ \delta_{(s,l)}$.□

Proof of Lemma 1 It follows from Lemmata 1 and the inclusion $\Delta \subset \cup_{s \in \underline{\mathbf{S}}} \ \{s\} \times P_s$. □

Lemma 3 *The following Assertions hold:*

(i) *the price sequences,* $\{\omega_0^n := (p_0^n, q^n)\}$ *and* $\{p_s^n\}$*, may be assumed to converge, say to* $\omega_0^* := (p_0^*, q^*) \in P_0$ *and* $p^* := (p_s^*) \in \mathcal{P} := P^{\underline{\mathbf{S}}}$*, s.t.* $\omega_s^* := (s, p_s^*) \in \overline{\Delta} \subset \cap_{i \in I} \Omega_i$*, for all* $s \in \underline{\mathbf{S}}$*;*

(ii) *the sequences* $\{(x_{is}^n)_{s \in \underline{\mathbf{S}}'}\}$ *and* $\{z_i^n\}$*, for each* $i \in I$*, may be assumed to converge, say to* $(x_{is}^*)_{s \in \underline{\mathbf{S}}'} \in \mathbb{R}_+^{L \times \underline{\mathbf{S}}'}$ *and* $z_i^* \in \mathbb{R}^J$*, such that* $\sum_{i \in I} \ (x_{is}^* - e_{is})_{s \in \underline{\mathbf{S}}'} = 0$*, and* $\sum_{i \in I} z_i^* = 0$.

Proof Assertion (i) From the definitions and Lemma 1, the relations $p_s^n \in [\delta, 1]^L \cap P$ and $(s, p_s^n) \in \Delta$ hold, for every $n \geqslant N$ along Lemma 2, and every $s \in \underline{\mathbf{S}}$. The sequence $\{p_s^n\}_{n \geqslant N}$ may, hence, be assumed to converge, say to $p_s^* \in P$, such that $(s, p_s^*) \in \overline{\Delta}$, for every $s \in \underline{\mathbf{S}}$. The relation $\overline{\Delta} \subset (\cap_{i \in I} \Omega_i)$ holds from Assumption *A3* and Definition 1. Similarly, the sequence $\{\omega_0^n := (p_0^n, q^n)\}$ may be assumed to converge in the compact set P_0, say to $\omega_0^* := (p_0^*, q^*) \in P_0$. Assertion (i) follows. □

Assertion (ii) The non-negativity and market clearance conditions over auxiliary equilibrium allocations imply that $\{(x_{is}^n)_{s \in \underline{\mathbf{S}}'}\}_{n \geqslant N}$ is bounded, hence, may be assumed to converge, for each $i \in I$, say to $(x_{is}^*)_{s \in \underline{\mathbf{S}}'} \in \mathbb{R}_+^{L \times \underline{\mathbf{S}}'}$. The market clearance relations, $\sum_{i \in I} \ (x_{is}^n - e_{is})_{s \in \underline{\mathbf{S}}'} = 0$, hold for each $n \geqslant N$, and yield, the limit: $\sum_{i \in I} \ (x_{is}^* - e_{is})_{s \in \underline{\mathbf{S}}'} = 0$.

For each $i \in I$, we let $Z_i := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in \Omega_i\}$, $Z_i^\perp$ be its orthogonal complement, and $z_i'^n$ be the orthogonal projection of z_i^n on $Z_i^\perp$, for each $n \geqslant N$. We also let $Z_o := \sum_{i \in I} Z_i$ and $Z := \{(z_i) \in \times_{i \in I} Z_i^\perp : \|(z_i)\| = 1\}$.

We now show that the sequence $\{(z_i'^n)\}$ is bounded. Assume, by contraposition, that there exists an extracted sub-sequence, $\{(z_i'^{\varphi(n)})\}_{n \geqslant N}$, of non-zero portfolios, such that $\lim_{n \to \infty} \|(z_i'^{\varphi(n)})\| = \infty$. Then, the portfolio collection, $(\overline{z}_i^n) := \frac{(z_i'^{\varphi(n)})}{\|(z_i'^{\varphi(n)})\|}$, belongs to Z, for every $n \geqslant N$, a compact set. Therefore the sequence, $\{(\overline{z}_i^n)\}$, may be assumed to converge, say to $(z_i) \in Z$. We let $\alpha := 1 + \|(e_i)\| > 0$ be given and observe that the following relations hold, from the definition of auxiliary equilibria:

$\sum_{i \in I} \overline{z}_i^n \in Z_o$ and $V(\omega) \cdot \overline{z}_i^n \geqslant \frac{-\alpha}{\|(z_i'^{\varphi(n)})\|}$, $\forall (i, \omega) \in I \times \Omega_i^n$, $\forall n \geqslant N$, and

$\sum_{i \in I} z_i \in Z_o$ and $V(\omega) \cdot z_i \geqslant 0$, $\forall (i, \omega) \in I \times \Omega_i$, when passing to the limit.

Since (Ω_i) meets the AFAO Condition, the latter relations imply $V(\omega) \cdot z_i = 0$ for every $(i, \omega) \in I \times \Omega_i$, hence, $z_i \in Z_i \cap Z_i^\perp = \{0\}$ for every $i \in I$, which contradicts the fact that $(z_i) \in Z$. This contradiction proves that $\{(z_i'^n)\}$ is bounded. Moreover, from the equilibrium relations $\sum_{i \in I} z_i'^n \in Z_o$ (for $n \geqslant N$), the sequence $\{(z_i^n)\}$ may be assumed to be bounded (since $\{(z_i'^n)\}$ is), hence, to converge, say to $\{(z_i^*)\}$. Then, the equilibrium relations, $\sum_{i \in I} z_i^n = 0$ (for $n \geqslant N$), yield, in the limit: $\sum_{i \in I} z_i^* = 0$. □

Lemma 4 *For all* $(i, \omega := (s, p), z) \in I \times \Omega_i \times \mathbb{R}^J$, *let* $B_i(\omega, z) = \{x \in \mathbb{R}_+^L : p \cdot (x - e_{is}) \leqslant V(\omega) \cdot z\}$. *Using the notations of Lemma 3, the following Assertions hold, for each* $i \in I$:

(i) $\cap_{\omega \in \Omega_i} B_i(\omega, z_i^*) \neq \varnothing$;

(ii) the correspondence $\omega \in \Omega_i \mapsto \arg\max \ u_i(x_{i0}^*, x)$, *for* $x \in B_i(\omega, z_i^*)$, *is a continuous map, whose embedding,* $x_i^* : \{0\} \cup \Omega_i \mapsto x_{i\omega}^*$, *is a consumption plan, i.e.,* $x_i^* \in X_{\pi_i}$;

(iii) $x_{i\omega_s^*}^* = x_{is}^*$, *for each* $s \in \underline{\mathbf{S}}$;

(iv) $U_i^{\pi_i}(x_i^*) = \lim_{n \to \infty} u_i^n(x_i^n)$.

Proof Let $i \in I$ be given throughout.

Assertion (i) From Definition 4, the relations, $-p_s^i \cdot e_{is} \leqslant p_s^i \cdot (x_\omega^n - e_{is}) \leqslant V(\omega) \cdot z_i^n$, hold, for every $n \geqslant N$ and every $\omega := (s, p_s^i) \in \Omega_i^n$, and imply the following one, from Lemmas 2 and 3 and the continuity of V: $0 \in \cap_{\omega \in \Omega_i} B_i(\omega, z_i^*)$. □

Assertion (ii) From Definition 4, Lemma 2 and Assumption *A2*, the following relations hold for each $n \geqslant N$ and every $\omega \in \Omega_i$: $\{x^n_{i\Phi_i^n(\omega)}\} = \arg\max_{x \in B_i(\Phi_i^n(\omega), z_i^n)} u_i(x_{i0}^n, x)$.

Let $Z := \{z \in \mathbb{R}^J : \|z\| \leqslant 2\|z_i^*\| + 1\}$ and $R := \{(\omega, z) \in \Omega_i \times Z : B_i(\omega, z) \neq \varnothing\}$ be given non-empty sets. The correspondence B_i is closed continuous and $B_i(R)$ is bounded from the definition of Ω_i. From Berge Theorem (Debreu, 1959, p. 19) and Assumption *A2*, $(x_0, \omega, z) \in \mathbb{R}_+^L \times R \mapsto \arg\max_{x \in B_i(\omega, z)} u_i(x_0, x)$ is a continuous map.

From Lemmas 2 and 3, the relations $(x_{i0}^*, \omega, z_i^*) = \lim_{n \to \infty}(x_{i0}^n, \Phi_i^n(\omega), z_i^n)$ hold for every $\omega \in \Omega_i$. From the continuity of $(x_0, \omega, z) \in \mathbb{R}_+^L \times R \mapsto \arg\max_{x \in B_i(\omega, z)} u_i(x_0, x)$, the relations, $\{x^n_{i\Phi_i^n(\omega)}\} = \arg\max_{x \in B_i(\Phi_i^n(\omega), z_i^n)} u_i(x_{i0}^n, x)$, for all $n \geqslant N$ and $\omega \in \Omega_i$, pass to the limit and yield a continuous map, $\omega \in \Omega_i \mapsto x_{i\omega}^* := \arg\max_{x \in B_i(\omega, z_i^*)} u_i(x_{i0}^*, x)$, whose embedding, $x_i^* : \omega \in \Omega_i' := \{0\} \cup \Omega_i \mapsto x_{i\omega}^*$, is a consumption plan of $\mathcal{E}_{(\pi_i)}$, i.e., $x_i^* \in X_i$.□

Assertion (iii) Let $s \in \underline{\mathbf{S}}$ be given. For every $n \geqslant N$ along Lemma 2, let $\omega_s^n := (s, p_s^n) \in \Omega_i$. Resuming the proof of Assertion (ii), above, the map, $(x_0, \omega, z) \in \mathbb{R}_+^L \times R \mapsto \arg\max_{x \in B_i(\omega, z)} u_i(x_0, x)$, is continuous. From Definition 4, the relations $x^n_{i\Phi_i^n(\omega_s^n)} = \arg\max_{x \in B_i(\Phi_i^n(\omega_s^n), z_i^n)} u_i(x_{i0}^n, x)$ and $x_{i\mathbf{s}}^n = \arg\max_{x \in B_i(\omega_s^n, z_i^n)} u_i(x_{i0}^n, x)$ hold, for every $n \geqslant N$. Passing to the limit, these relations yield, from Lemma 3 and above: $x^*_{i\omega_s^*} = \lim_{n\to\infty} x^n_{i\Phi_i^n(\omega_s^n)} = \arg\max_{x \in B_i(\omega_s^*, z_i^*)} u_i(x_{i0}^*, x) = \lim_{n \to \infty} x_{is}^n = x_{is}^*$. □

Assertion (iv) Let $x_i^* \in X_{\pi_i}$ be defined from above. Resuming the proof of Assertion (ii), we let $\varphi_i : (x_0, \omega, z) \in \mathbb{R}_+^L \times R \mapsto \arg\max_{x \in B_i(\omega, z)} u_i(x_0, x)$ be a continuous map. The continuity of $U_i : (x_0, \omega, z) \in \mathbb{R}_+^L \times R \mapsto u_i(x_0, \varphi_i(x_0, \omega, z))$ follows from that of u_i and φ_i.

We recall that the relations $(x^*_{i0}, \omega, z^*_i) = \lim_{n\to\infty}(x^n_{i0}, \Phi^n_i(\omega), z^n_i)$, $u_i(x^*_{i0}, x^*_{i\omega}) = U_i(x^*_{i0}, \omega, z^*_i)$ and $u_i(x^n_{i0}, x^n_{i\Phi^n_i(\omega)}) = U_i(x^n_{i0}, \Phi^n_i(\omega), z^n_i)$ hold, from Lemmas 2 and 3 and above, for every $\omega \in \Omega_i$ and every $n \geqslant N$. We also recall that $B_i(R)$ is bounded. Then, the following relations follow from above, the uniform continuity of u_i and U_i on compact sets, Lemma 2 and the definitions of the utility functions:

(I) $\quad \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} : \forall n > N_\varepsilon,\ \forall \omega \in \Omega_i,\ \mid u_i(x^*_{i0}, x^*_{i\omega}) - u_i(x^n_{i0}, x^n_{i\Phi^n_i(\omega)}) \mid\ < \varepsilon.$

(II) $\quad U^{\pi_i}_i(x^*_i) := \int_{\omega\in\Omega_i} u_i(x^*_{i0}, x^*_{i\omega}) d\pi_i(\omega);$

(III) $\quad u^n_i(x^n) := \frac{1}{n\#\underline{\mathbf{S}}} \sum_{s\in\underline{\mathbf{S}}} u_i(x^n_0, x^n_s) + (1 - \frac{1}{n}) \sum_{\omega\in\Omega^n_i} u_i(x^n_0, x^n_\omega)\pi^n_i(\omega)$, for each $n \geqslant N$.

Assertion (iv) results immediately from relations (I), (II) and (III), above. □

References

[1] Arrow, K.J., Le rôle des valeurs boursières pour la répartition la meilleure des risques, Econometrie CNRS 40, 41-47, 1953.

[2] Cass, D., Competitive equilibrium with incomplete financial markets, CARESS Working Paper 84-09, University of Pennsylvania, 1984.

[3] Cornet, B., De Boisdeffre, L., Arbitrage and price revelation with asymmetric information and incomplete markets, Journal of Math. Economics 38, 393-410, 2002.

[4] Cornet, B., De Boisdeffre, L., Elimination of arbitrage states in asymmetric information models, Economic Theory 38, 287-293, 2009.

[5] De Boisdeffre, L., No-arbitrage equilibria with differential information: an existence proof, Economic Theory 31, 255-269, 2007.

[6] De Boisdeffre, L., Learning from arbitrage, Econ Theory Bull 4, 111-119, 2016.

[7] De Boisdeffre, L., Equilibrium in incomplete markets with Differential Information: A basic Model of Generic Existence, Math. Soc. Sciences, 2021, forthcoming.

[8] Debreu, G., Theory of Value, Yale University Press, New Haven, 1959.

[9] Duffie, D., Shafer, W., Equilibrium in Incomplete Markets: A basic Model of Generic Existence, Equilibrium in incomplete markets, A basic Model of Generic Existence, J. Math. Econ. 14, 285-300, 1985.

[10] Grandmont, J.-M., Temporary general equilibrium theory, in Handbook of Math. Econ., K.J. Arrow & M.D. Intriligator Eds, North-Holland, 879-922, 1982.

[11] Hart, O., On the optimality of equilibrium when the market structure is incomplete, JET 11, 418-433, 1975.

[12] Kurz, M., On rational belief equilibria, Economic Theory 4, 859-876, 1994.

[13] Kurz, M., On the structure and diversity of rational beliefs, Economic Theory 4, 877-900, 1994.

[14] Kurz, M., Wu, H.M., Endogenous uncertainty in a general equilibrium model with price contingent contracts, Economic Theory 8, 461-488, 1996.

[15] Radner, R., Existence of equilibrium plans, prices and price expectations in a sequence of markets. Econometrica 40, 289-303, 1972.

[16] Radner, R., Rational expectations equilibrium: generic existence and the information revealed by prices. Econometrica 47, 655-678, 1979.

[17] Radner, R., Equilibrium under uncertainty, in Handbook of Math. Econ., K.J Arrrow & M.D. Intriligator Eds, North Holland, 923-1002, 1982.

[18] Starr, R., The new palgrave finance: a dictionary of economics, J. Eatwell, M. Milgate, P. Neuman Eds, Macmillan, 1989.