



**Subjective risk and disappointment**

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ABSTRACT: If an investor does care for utilities –and not for monetary outcomes– stochastic dominances should be expressed in terms of utility units ("utils"). If so, any "rational" investor may be characterized by an elementary utility function –called canonical utility function– which is such that the partial weak order induced by stochastic dominance over utils is as "close" to the weak order of preferences as possible. As a consequence, the random utilities of the available prospects do not violate the second-order stochastic dominance property. Substituting utils for monetary units leads to substitute "subjective" risk for "objective" risk *à la* Rothschild and Stiglitz (1970). A weakened independence axiom may then be set over comparable prospects, *i.e.* those which exhibit the same canonical expected utility. This leads to a fully choice-based theory of disappointment. The functional is lottery-dependent (Becker and Sarin 1987). When constant marginal utility is assumed, it is but the opposite to a convex measure of risk (Föllmer and Schied 2002). It may be viewed as a theoretical justification for choosing this measure of risk.

JEL classification: D81. Key-words: disappointment, risk-aversion, subjective risk, risk premium, expected utility.

RESUME : Si un agent économique rationnel est moins sensible au niveau de son revenu qu'à l'utilité de celui-ci, effectuer des tests de dominance stochastique n'a de sens que si ces tests portent sur les utilités de revenus aléatoires. On peut alors montrer que cet agent est caractérisé par une fonction d'utilité élémentaire telle que les utilités des revenus aléatoires ne violent jamais la dominance stochastique de second ordre. Le préordre induit par la dominance stochastique est alors aussi proche que possible du préordre des préférences et l'on peut généraliser la notion de risque à la Rothschild et Stiglitz en raisonnant en terme de risque "subjectif" où les valeurs sont exprimées en utilités. On peut enfin poser un axiome d'indépendance affaibli qui n'est valide que pour les revenus aléatoires "comparables" c'est-à-dire ceux qui ont la même espérance d'utilité. Cette nouvelle axiomatisation du comportement d'un individu en univers incertain aboutit à une théorie de la déception où la fonctionnelle représentant les préférences est "loterie-dépendante" (Becker and Sarin 1987). Si l'utilité marginale de l'investisseur est constante, la nouvelle fonctionnelle n'est que l'opposé d'une mesure de risque convexe (Föllmer and Schied 2002) et elle peut constituer la justification de celle-ci.

JEL classification: D81. Mots-clés : déception, aversion pour le risque, risque subjectif, prime de risque, utilité espérée.

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# 1 Introduction

According to many theories of decision making under risk, preferences may be represented by a functional which depends on the utility of a random outcome. This means that an investor is sensitive but to the utility of his wealth. Equivalently, one can say that, to take a financial decision, he only needs to know the values of the utilities of the outcomes he may get. A well-known example is the case when expected utility theory (henceforth EU theory) is valid: the value of a random prospect is a probability weighted average of the utilities of the possible outcomes. When marginal utility is variable, an investor is risk-averse (prone) if his elementary utility function is concave (convex). In the particular case of constant marginal utility, he is risk-neutral. Finally, in any case, the investor takes into account but an average of the results of a gamble to which he is sensitive. In other words, whatever his attitude towards risk (risk-aversion, risk-proneness or neutrality), he behaves in the same way. This looks as a paradox. A second paradox is that checking for the rationality of the behaviour of an investor is often undertaken through implementing a second-order stochastic dominance test where monetary values are taken into account. This is clearly inconsistent if the investor cares for utilities rather than for monetary values.

By contrast, if an investor behaves according to a theory such as that of Loomes and Sugden (henceforth L&S) (1986), he is considered as risk-neutral as long as he averages utilities. He is risk-averse/prone if and only if (henceforth iff) his welfare includes expected elation/disappointment in addition to the expected utility of his wealth. Elation/disappointment depends on the gap between the utility of the actual outcome and the expected utility of the prospect. Moreover, it becomes easy to define stochastic dominance with respect to "utils" —i.e. units of welfare— instead of monetary units using the frame of L&S (1986). Risk may then be defined in accordance with the new definition of second-order stochastic dominance.

Unfortunately, the theory of L&S (1986) lacks an axiomatic basis, and so do most of the other disappointment models developed since that time. As a consequence, it seems of interest to develop a fully choice-based theory of disappointment which will allow for a satisfactory distinction between risk-averse, risk-neutral and risk-prone investors.

The main idea of this paper is that the utility function of an investor may be inferred from a comparison between the weak order which represents his preferences and the partial preorder induced by second-order stochastic dominance over utils. The "closer" to the former is the latter, the more likely the utility function is to represent preferences. Of course, to make sense, the previous statement requires that the closeness between the two weak orders will have first been defined —which will be done in Section 2—.

Next, a partition of the set of the random prospects will be made. Each subset then includes prospects exhibiting the same expected utility. The independence axiom will be set over each subset of prospects and a standard representation theorem will be used to define, for any subset, a lottery-dependent functional. Finally, preferences will be represented by a functional which encom-

passes, as particular cases, that of EU theory and that of Loomes and Sugden. Unlike most of its predecessors, this lottery-dependent model preserves a fully choice-based approach and is testable. Moreover, the corresponding decision theory is endowed with two interesting properties: (i) when constant marginal utility is assumed, the certainty equivalent of a prospect is but the opposite to a convex measure of risk *à la* Föllmer and Schied (2002) and (ii) it provides a mathematical expression of the value of a risk premium as a function of risk aversion and of a quantity of risk. However, to get the last property, one must leave aside the usual definition of risk –which is a scalar– and split any risk premium into elementary risk premia, each of which may be identified to the product of a quantity of risk by risk aversion.

The rest of this article is organized as follows: first, stochastic dominance is revisited and the definitions of subjective risk and of rational investors are clarified. Next, the axiomatization of a general theory of disappointment is developed. Section 4 concludes.

## 2 Revisiting stochastic dominance

### 2.1 Preliminary definitions

From now on, we consider a set of random prospects, labelled  $\mathbb{W}$ , whose outcomes are monetary and belong to a bounded interval of  $\mathbb{R}$ , say  $[a, b]$ . An element of  $\mathbb{W}$  will be labelled  $\tilde{w}$  and its cumulative distribution function  $F_{\tilde{w}}(\cdot)$ . If a random prospect  $\tilde{w}$  has a discrete support  $\{w_1, w_2, \dots, w_K\}$ , it will also be denominated  $[w_1, w_2, \dots, w_K ; p_1, p_2, \dots, p_K]$  where  $p_k = \Pr(\tilde{w} = w_k)$ . A probability mixture of  $\tilde{w}_1$  and  $\tilde{w}_2$ , will be denoted  $\alpha\tilde{w}_1 \oplus (1 - \alpha)\tilde{w}_2$ , where  $\alpha$  belongs to  $[0, 1]$ . The degenerate lottery whose outcome is  $w$  with certainty is  $\delta(w)$ . Preferences over prospects will be denoted  $\succsim$ , with  $\prec$  (strict preference) and  $\sim$  (indifference). The certainty equivalent of the prospect  $\tilde{w} \in \mathbb{W}$  is labelled  $\mathbf{c}(\tilde{w})$ , i.e.  $\tilde{w} \sim \delta(\mathbf{c}(\tilde{w}))$ . A normalized elementary utility function (henceforth *n.e.u.* function) is a continuously derivable and strictly increasing function mapping  $[a, b]$  on to  $[0, 1]$ . The set of *n.e.u.* functions will be denoted  $\mathbb{U}$ .

Simultaneously, partial weak orders may be defined, independently of preferences: they include first and second-order stochastic dominance (henceforth *FSD* and *SSD*). *FSD* (*SSD*) will be denoted  $\succsim_1$  ( $\succsim_2$ ) with  $\prec_1$  ( $\prec_2$ ) for strict dominance. A partial weak order induced by *FSD* (*SSD*) is consistent with the total weak order of preferences if we have the following implication:  $\tilde{w}_1 \succsim_1 \tilde{w}_2 \Rightarrow \tilde{w}_1 \preceq \tilde{w}_2$  ( $\tilde{w}_1 \succsim_2 \tilde{w}_2 \Rightarrow \tilde{w}_1 \preceq \tilde{w}_2$ ). It is often argued that a "good" theory of decision making under risk must be endowed with two properties: (*Property a*) For any investor, the weak order induced by first-order stochastic dominance is consistent with the weak order of preferences. (*Property b*) For any risk-averse investor, the weak order of preferences is consistent with the weak order induced by second-order stochastic dominance. As already said, this way of reasoning is somewhat paradoxical since, on the one hand, an investor is assumed to be sensitive to the utility of an outcome but, on the other hand, the consistency

of his behaviour is checked with a test of stochastic dominance over monetary outcomes. Hence it seems preferable to substitute utils for monetary values *i.e.* to define subjective stochastic dominances as indicated below:

**Definition 1 (First-order and second-order subjective stochastic dominance (henceforth FSSD and SSSD)).** Let  $\tilde{w}_1$  and  $\tilde{w}_2$  be two arbitrary random prospects. Let  $u(\cdot)$  be a *n.e.u.* function and let  $\tilde{\omega}_i = u(\tilde{w}_i)$  for  $i = 1, 2$ . It is equivalent to state that  $\tilde{w}_1$  dominates  $\tilde{w}_2$  by FSSD (SSSD) or that  $\tilde{\omega}_1$  dominates  $\tilde{\omega}_2$  by FSD (SSD), *i.e.*  $\tilde{w}_1 \lesssim_1^u \tilde{w}_2 \Leftrightarrow \tilde{\omega}_1 \lesssim_1 \tilde{\omega}_2$  ( $\tilde{w}_2 \gtrsim_2^u \tilde{w}_1 \Leftrightarrow \tilde{\omega}_2 \gtrsim_2 \tilde{\omega}_1$ ), where FSSD (SSSD) is denoted  $\lesssim_1^u$  ( $\gtrsim_2^u$ ).

Of course, looking at levels of outcomes may be equivalent to looking at utilities. This happens to be the case when first-order dominance is taken into account. Indeed, FSD is a property which is conservative through the change of random variable:  $\tilde{\omega} = u(\tilde{w})$ .<sup>1</sup> By contrast, this result is no longer valid, when second-order stochastic dominance is considered. Actually, the following characterization of SSSD holds:

**Proposition 1 (characterization of SSSD).** Let  $\tilde{w}_1$  and  $\tilde{w}_2$  be two arbitrary random prospects, let  $u(\cdot)$  be a *n.e.u.* function and let  $\tilde{\omega}_i = u(\tilde{w}_i)$  for  $i = 1, 2$ . It is equivalent to state: (a)  $\tilde{w}_1$  dominates  $\tilde{w}_2$  by SSSD (*i.e.*  $\tilde{w}_2 \gtrsim_2^u \tilde{w}_1$ ) or (b)  $\int_a^z u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \leq 0$  for any  $z \in [a, b]$

**Proof.** It is given in the Appendix.  $\square$ .

As a consequence, it is convenient to set the following definition

**Definition 2.** A *n.e.u.* function  $u(\cdot)$  is consistent with the weak order of preferences  $\preceq$  if the partial weak order induced by SSSD never contradicts the total weak order induced by preferences, *i.e.* if, for any pair of prospects  $(\tilde{w}_1, \tilde{w}_2)$  such that  $\tilde{w}_2 \gtrsim_2^u \tilde{w}_1$ , then  $\tilde{w}_2 \succsim \tilde{w}_1$ .

By contrast, if there exists at least one pair of prospects  $(\tilde{w}_1, \tilde{w}_2)$  such that either  $\tilde{w}_2 \gtrsim_2^u \tilde{w}_1$  and  $\tilde{w}_2 \succ \tilde{w}_1$ , or such that  $\tilde{w}_2 \prec_2^u \tilde{w}_1$  and  $\tilde{w}_2 \gtrsim \tilde{w}_1$ , then, the *n.e.u.* function  $u(\cdot)$  is not consistent with the weak order of preferences  $\preceq$ . If an investor cares for utilities, his *n.e.u.* function will be consistent with his weak order of preferences and it becomes of interest to consider a "subjective" notion of risk whose definition will be consistent with SSSD –*i.e.* with SSD over utilities–. Actually, the definition given by Rothchild and Stiglitz (henceforth R&S) (1970) for "objective" risks may be transposed in the following way:

**Definition 3 (subjective risk).** Let  $u(\cdot)$  be a *n.e.u.* function. The random prospect  $\tilde{w}_1$  is subjectively less risky than the random prospect  $\tilde{w}_2$  if the two following conditions are met:

- (a)  $\tilde{w}_1$  dominates  $\tilde{w}_2$  by SSSD (*i.e.*  $\tilde{w}_2 \gtrsim_2^u \tilde{w}_1$ ) and
- (b)  $\int_a^b u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx = 0$ .

Note that SSSD has been substituted for SSD in condition (a) whereas the integral condition (b) coincides with that of R&S (1970) if marginal utility is constant. As a consequence, if  $\tilde{w}_1$  is subjectively less risky than  $\tilde{w}_2$ , then its expected utility is equal to that of  $\tilde{w}_2$ . Indeed, we have the following equality:

$$\int_a^b u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx = -(\mathbb{E}[u(\tilde{w}_1)] - \mathbb{E}[u(\tilde{w}_2)]) \quad (1)$$

<sup>1</sup>The proof of this statement is trivial.

We now give another example: consider the preorder of preferences which is represented by the following functional:

$$\mathbf{U}(\tilde{w}) \stackrel{def}{=} \int_a^b (u(x) + \mathcal{E}(u(x) - \mathbb{E}[u(\tilde{w})])) dF_{\tilde{w}}(x) \quad (2).$$

where the following conditions are met: (a)  $\mathcal{E}(0) = 0$ , (b)  $0 < \mathcal{E}'(x) < 1$ , (c)  $\mathcal{E}''(x) < 0$  and (d)  $\sup \mathcal{E}'(z) \leq 1$ .<sup>2</sup> The functional (2) has been introduced by L&S (1986) and it will be called, from now on, a *LS-functional*. As shown in the next proposition, the elementary utility function of a LS-functional is consistent.

**Proposition 2.** *Let  $u(\cdot)$  be the n.e.u. function of a model where preferences are represented by a LS-functional: then  $u(\cdot)$  is consistent with the weak order of preferences represented by the functional.*

**Proof.** It is given in the Appendix.  $\square$

## 2.2 Properties of consistent utility functions

Recall that the weak order induced by *FSSD* (*SSSD*) is partial. Hence the following definition will make sense.

**Definition 4 (comparable prospects).** *Two prospects are comparable with respect to  $u(\cdot)$  –or, in short, comparable–, iff either  $\tilde{w}_1$  dominates  $\tilde{w}_2$  by *SSSD* or if  $\tilde{w}_2$  dominates  $\tilde{w}_1$  by *SSSD*. The subset of prospects which are comparable with respect to  $u(\cdot)$  will be denominated  $\mathbf{W}_2^u$ .*

Let  $\mathbf{W}_2^{u+}$  ( $\mathbf{W}_2^{u-}$ ) consist in the subset of pairs of prospects  $(\tilde{w}_1, \tilde{w}_2)$  over which the two weak orders,  $\preceq_2^u$  and  $\preceq$ , coincide (disagree). Clearly, the two subsets constitute a partition of  $\mathbf{W}_2^u$  and the n.e.u. function  $u(\cdot)$  is all the more a good candidate for representing the preferences of an investor that  $\mathbf{W}_2^{u+}$  is larger and  $\mathbf{W}_2^{u-}$  more tiny.

It is easy to increase the size of  $\mathbf{W}_2^{u+}$  through making  $u(\cdot)$  more and more concave. Unfortunately, such a process leads to an increase of the size of the subset of comparable prospects and, consequently, the size of  $\mathbf{W}_2^{u-}$  may also increase... This dilemma is detailed in the below proposition and corollaries.

**Proposition 3.** *Let  $u(\cdot)$  and  $v(\cdot)$  be two n.e.u. functions such that  $u(\cdot)$  is more concave than  $v(\cdot)$ . Then, we have the following implication:*

$$\tilde{w}_1 \preceq_2^v \tilde{w}_2 \Rightarrow \tilde{w}_1 \preceq_2^u \tilde{w}_2$$

and the following inclusions:

$$\mathbf{W}_2^{v+} \subseteq \mathbf{W}_2^{u+} ; \mathbf{W}_2^{v-} \subseteq \mathbf{W}_2^{u-}$$

**Proof.** It is given in the Appendix.  $\square$

Actually, we want to describe the behaviour of a rational investor, i.e. we want to rule out violations of *SSSD*. In other words, we are looking for a n.e.u. function which will be consistent. Actually, many n.e.u. functions are consistent. Hence, we must choose among them the "best" one. To do this, one may define the closeness of two weak orders over  $\mathbb{W}$  as follows: the weak order  $\preceq_2^u$  is closer to the weak order  $\preceq$  than the weak order  $\preceq_2^v$  iff either

<sup>2</sup> The reasons for these conditions are: (a) no elation/disappointment is experimented if the actual outcome coincides with its expected value; (b and c) elation/disappointment is an increasing and concave function of the difference between the utility of the actual outcome and that of its expected value; (d) disappointment must not vary too quickly.

$$\mathbf{W}_2^{u-} \subset \mathbf{W}_2^{v-} \text{ or } \mathbf{W}_2^{u-} = \mathbf{W}_2^{v-} \text{ and } \mathbf{W}_2^{v+} \subseteq \mathbf{W}_2^{u+}.$$

In the case when two consistent *n.e.u.* functions,  $u(\cdot)$  and  $v(\cdot)$ , are considered, we have  $\mathbf{W}_2^{u-} = \mathbf{W}_2^{v-} = \emptyset$  and  $u(\cdot)$  is "better" than  $v(\cdot)$  -i.e.  $\lesssim_2^u$  is closer to  $\preceq$  than  $\lesssim_2^v$  - iff  $\mathbf{W}_2^{v+} \subseteq \mathbf{W}_2^{u+}$ . Of course, one cannot rule out, on *a priori* grounds, that there may exist two consistent *n.e.u.* functions  $u(\cdot)$  and  $v(\cdot)$  such that neither  $\mathbf{W}_2^{u+} \subset \mathbf{W}_2^{v+}$  nor  $\mathbf{W}_2^{v+} \subset \mathbf{W}_2^{u+}$ .

However, if we focus on functions which are *either convex or concave*, it can be shown that there exists a *n.e.u.* function,  $\mathbf{u}(\cdot)$ , which is such that the weak order  $\lesssim_2^{\mathbf{u}}$  is, among the weak orders induced by *SSSD*, the closest to the weak order of preferences. Indeed, let  $\mathbb{U}_C \subset \mathbb{U}$  denominate the subset of *n.e.u.* functions which are either concave or convex and  $\mathbb{U}_C^*$  ( $\bar{\mathbb{U}}_C$ ) the subset of elements of  $\mathbb{U}_C$  which are consistent (inconsistent). As shown in the next proposition, there exists a consistent *n.e.u.* function which is the "best" among the elements of  $\mathbb{U}_C$ .

**Proposition 4.**

(a) *If no violation of SSD occurs, then, the lower envelope of the functions belonging to  $\mathbb{U}_C$  is consistent and it is the most concave among the consistent and concave n.e.u. functions.*

(b) *If violations of SSD occur, then, the higher envelope of the functions belonging to  $\mathbb{U}_C^*$  is consistent and it is the less convex among the consistent and convex n.e.u. functions.*

(c) *In both cases, the envelope induces a weak order  $\lesssim_2^{\mathbf{u}}$  which is the closest to the weak order of preferences among the partial orders induced by concave or convex n.e.u. functions.*

**Proof.** It is given in the Appendix.  $\square$

Indeed, two cases may occur: either some violations of *SSD* exist or not. In the latter case, it can be shown that  $\mathbb{U}_C$  contains convex and concave consistent *n.e.u.* functions whereas, in the former one, it contains only convex functions. As a consequence, we may now set the following definition:

**Definition 5 (rationalizing functions).** *If violations of SSD occur (do not occur), the higher (lower) envelope of the consistent n. e .u. functions which are convex (concave), is called a rationalizing function, or, equivalently, one can say that it rationalizes the weak order of preferences.*

Examples where a standard utility function rationalizes the preferences of an investor have been already provided. This is a strong incentive for developing a theory of decision making under risk in which each investor will be endowed with a rationalizing utility function.

### 3 A simple theory of decision making under risk

A fully choice-based theory of decision making under risk is now presented. To allevy the exposition, *we focus on the case when no violations of SSD occur*. As a consequence, if preferences are rationalized by a *n.e.u.* function, then the function is concave.

### 3.1 The axiomatics

The first step consists in assuming that preferences obey the two first axioms of EU theory.

**Axiom 1 (total ordering of  $\preceq$ ).** *The binary relation  $\preceq$  is a complete weak order.*

**Axiom 2 (continuity of  $\preceq$ ).** *For any prospect  $\tilde{w} \in \mathbb{W}$  the sets  $\{\tilde{v} \in \mathbb{W} \mid \tilde{v} \preceq \tilde{w}\}$  and  $\{\tilde{v} \in \mathbb{W} \mid \tilde{w} \preceq \tilde{v}\}$  are closed in the topology of weak convergence.*

Axioms 1 and 2 imply the Archimedean property according to which, if  $\tilde{w}'' \preceq \tilde{w} \preceq \tilde{w}'$ , then there exists  $\alpha \in [0, 1]$  such that  $\alpha \tilde{w}' \oplus (1 - \alpha) \tilde{w}'' \sim \tilde{w}$ . They also imply that there exists a continuous utility functional,  $\mathcal{U}(\cdot)$ , mapping  $\mathbb{W}$  on to an interval of  $\mathbb{R}$  which represents the investor's preferences. It is defined up to a strictly continuous and increasing transformation. From now on, the set which includes the functionals such as  $\mathcal{U}(\cdot)$  will be denominated  $\mathbb{V}$ .

To get stronger results, one (or more) additional axiom(s) must be set. In EU theory, a third axiom, namely the independence axiom is set. It is valid over the whole set  $\mathbb{W}$ . However, to account for anomalies, one must weaken the axiom. This will now be done through setting it only for prospects which are comparable. If there exists a *n.e.u.* function  $\mathbf{u}(\cdot)$  which rationalizes the preferences of the investor, then comparable prospects will exhibit the same expected utility. Hence, we set the following two axioms:

**Axiom 3 (rationality).** *Any investor is rational, i.e. investors' preferences are rationalized by a *n.e.u.* function  $\mathbf{u}(\cdot)$ , which will be called, from now on, the canonical utility function.*

**Axiom 4 (weak independence axiom).** *The independence property is met over each subset of comparable prospects.*

A well-known consequence of Axiom 4 is the following proposition:

**Proposition 5.** *Under Axioms 1 to 4, the weak order of preferences  $\preceq$  may be represented over  $\mathbb{W}_\pi$  by the lottery-dependent functional:*

$$\mathbf{U}_\pi(\tilde{w}) \stackrel{def}{=} \int_a^b \mathbf{v}_\pi(x) dF_{\tilde{w}}(x),$$
where  $\pi = \mathbb{E}[\mathbf{u}(\tilde{w})]$  and where  $\mathbf{v}_\pi(\cdot)$  is a continuous and increasing function mapping  $[a, b]$  on to  $[\mathbf{v}_\pi(a), \mathbf{v}_\pi(b)]$  which is defined up to an affine and positive transformation.

**Proof.** See, for instance, Fishburn (1970).  $\square$

From now on, we set the following normalization conditions:

$$\mathbf{v}_\pi(\mathbf{u}^{-1}(\pi)) = \pi \text{ and } \pi \mathbf{v}_\pi(b) + (1 - \pi) \mathbf{v}_\pi(a) = \mathbf{u}_\pi(\mathbf{3}),$$

where  $\mathbf{u}_\pi \stackrel{def}{=} \mathbf{u}(\mathbf{c}_\pi)$  and  $\mathbf{c}_\pi \stackrel{def}{=} \mathbf{c}(\tilde{w}_\pi^{a,b})$ .

As a consequence,  $\mathbf{v}_\pi(\cdot)$  is, from now on, unambiguously defined. Note that  $\mathbf{U}_\pi(\tilde{w}_\pi^{a,b}) = \mathbf{u}_\pi$  and that  $\mathbf{U}_\pi(\delta(\mathbf{u}^{-1}(\pi))) = \pi$ . Clearly, any random prospect  $\tilde{w} \in \mathbb{W}_\pi$  is such that  $0 \leq \mathbf{U}_\pi(\tilde{w}) \leq 1$  or, equivalently such that  $\tilde{w}_\pi^{a,b} \preceq \tilde{w} \preceq \delta(\mathbf{u}^{-1}(\pi))$  (4).

We now turn to some other important consequences of the above set of axioms.

For any  $\tilde{w} \in \mathbb{W}_\pi$ , there exists a unique real number  $\alpha_{\tilde{w}} \in [0, 1]$  such that  $\tilde{w} \sim \mathcal{L}_\pi(\alpha_{\tilde{w}})$ , or, equivalently, such that  $\mathbf{c}(\tilde{w}) = \mathbf{c}_\pi^{\alpha_{\tilde{w}}}$ . Its existence is a consequence

of Axiom 2. Moreover,  $\alpha_{\tilde{w}}$  is well unique since (a)  $\mathcal{L}_\pi(\alpha')$  strictly dominates  $\mathcal{L}_\pi(\alpha'')$  by *SSD* iff  $\alpha' > \alpha''$  and that (b), by assumption, *SSD* is never violated. Finally, the lottery-dependent functional  $\mathbf{U}_\pi(\cdot)$  expresses as:

$$\mathbf{U}_\pi(\tilde{w}) = \alpha_{\tilde{w}}\pi + (1 - \alpha_{\tilde{w}})\mathbf{u}_\pi,$$

and we may substitute  $\mathcal{L}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}(\alpha_{\tilde{w}})$  for  $\tilde{w}$  when ranking prospects.

Now, from (4) we get that the certainty equivalent of a random prospect  $\tilde{w} \in \mathbb{W}_\pi$  –or, equivalently, that of  $\mathcal{L}_\pi(\alpha_{\tilde{w}})$ –, lies between  $\mathbf{c}_\pi$  and  $\mathbf{u}^{-1}(\pi)$ . As a consequence, its utility  $\mathbf{u}(\mathbf{c}(\tilde{w}))$  –which is also  $\mathbf{u}(\mathbf{c}(\mathcal{L}_\pi(\alpha_{\tilde{w}})))$ – is a convex combination of  $\mathbf{u}_\pi$  and  $\pi$ . Hence we have:

$$\mathbf{u}(\mathbf{c}(\tilde{w})) = \beta_\pi\pi + (1 - \beta_\pi)\mathbf{u}_\pi$$

Clearly the real number  $\beta_\pi$  is a continuous and strictly increasing function of  $\alpha_{\tilde{w}}$  mapping  $[0, 1]$  on to itself. It will be denominated, from now on,  $\beta_\pi(\alpha_{\tilde{w}})$ . Let:

$$\mathbf{U}(\tilde{w}) \stackrel{\text{def}}{=} \beta_\pi(\alpha_{\tilde{w}})\pi + (1 - \beta_\pi(\alpha_{\tilde{w}}))\mathbf{u}_\pi = (\pi - \mathbf{u}_\pi)\beta_\pi(\alpha_{\tilde{w}}) + \mathbf{u}_\pi$$

The above functional clearly represents preferences over the whole set  $\mathbb{W}$  and its restriction over the subset of the degenerate lotteries coincides with the canonical utility function. Two different functionals cannot coincide on the subset of degenerate lotteries since they are defined up to a continuous and strictly increasing transformation. Hence,  $\mathbf{U}(\cdot)$  is the unique preference representing functional satisfying the following condition:  $\mathbf{U}(\delta(z)) = \mathbf{u}(z)$  for  $z \in [a, b]$ . The above discussion may be summed up in the following proposition.

**Proposition 6.** *Under Axioms 1 to 4, there exists a unique family of functions  $\{\beta_\pi(\cdot)\}_{\pi \in [0,1]}$  such that*

- (i) *each element of the family is a continuous and strictly increasing function mapping  $[0, 1]$  on to itself*
- (ii) *preferences of a rational investor over  $\mathbb{W}$  may be represented by the lottery-dependent functional:*

$$\mathbf{U}(\tilde{w}) = (\mathbb{E}[\mathbf{u}(\tilde{w})] - \mathbf{u}_{\mathbb{E}[\mathbf{u}(\tilde{w})]})\beta_{\mathbb{E}[\mathbf{u}(\tilde{w})]} \left( \frac{\int_a^b \mathbf{v}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}(x) dF_{\tilde{w}}(x) - \mathbf{u}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}}{\mathbb{E}[\mathbf{u}(\tilde{w})] - \mathbf{u}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}} \right) + \mathbf{u}_{\mathbb{E}[\mathbf{u}(\tilde{w})]} \quad (5)$$

where  $\tilde{w} \in \mathbb{W}$ ,  $\mathbf{u}(\cdot)$  is the canonical utility function,  $\mathbf{u}_{\mathbb{E}[\mathbf{u}(\tilde{w})]} \stackrel{\text{def}}{=} \mathbf{u}(\mathbf{c}(\tilde{w}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}^{a,b}))$ , and  $\mathbf{v}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}(\cdot)$  is defined according to Proposition 5 and to (3).

- (iii) *the restriction of  $\mathbf{U}(\tilde{w})$  over the subset of degenerate lotteries coincides with the canonical utility function or, equivalently,  $\mathbf{U}(\delta(z)) = \mathbf{u}(z)$ , for any  $z \in [a, b]$ .*

**Proof.** See the above discussion.  $\square$

The lottery-dependent functional in (5), is far too general to be implemented. Hence we can try to particularize it.

### 3.2 Regular preferences

A case of interest is when  $\beta_\pi$  is the identity function  $\beta_\pi(\alpha) = \alpha$ . If this property is shared by all the  $\beta_\pi(\cdot)$ 's, we shall say that *preferences are regular*. Clearly, the condition  $\beta_\pi(\alpha) = \alpha$  is equivalent to the following one:

$$\tilde{w}_\pi^{a,b} \sim \mathcal{L}_{\alpha\pi+(1-\alpha)\mathbf{u}_\pi}(\alpha).$$

Indeed, we then have:

$$\mathbf{U}(\mathcal{L}_{\alpha\pi+(1-\alpha)\mathbf{u}_\pi}(\alpha)) = (\pi - \mathbf{u}_\pi)\beta_{\alpha\pi+(1-\alpha)\mathbf{u}_\pi}(\alpha) + \mathbf{u}_\pi = \alpha(\pi - \mathbf{u}_\pi) + \mathbf{u}_\pi = \mathbf{U}(\tilde{w}_\pi^{a,b}).$$

Hence, preferences are regular if, given two arbitrary real numbers  $\pi \in [0, 1]$  and  $\alpha \in [0, 1]$ , the random prospect  $\tilde{w}_\pi^{a,b}$  is indifferent to  $\mathcal{L}_{\alpha\pi+(1-\alpha)\mathbf{u}_\pi}(\alpha)$ . In other words, the utility of the certainty equivalent of any random prospect  $\tilde{w} \in \mathbb{W}_\pi$  is a convex combination of  $\pi$  and  $\mathbf{u}_\pi$  whose weights coincide with those of the compound lottery  $\mathcal{L}_\pi(\alpha_{\tilde{w}})$  which is indifferent to  $\tilde{w}$ . Hence, we get the following corollaries:

**Corollary 1.** *Under Axioms 1 to 4, preferences are regular if and only if, for any couple of real numbers  $(\pi, \alpha) \in [0, 1] \times [0, 1]$ , the random prospect  $\tilde{w}_\pi^{a,b}$  is indifferent to  $\mathcal{L}_{\alpha\pi+(1-\alpha)\mathbf{u}(\mathbf{c}(\tilde{w}_\pi^{a,b}))}(\alpha)$ .*

**Proof.** See the above discussion.  $\square$

**Corollary 2.** *Under Axioms 1 to 4, regular preferences  $\preceq$  may be represented over  $\mathbb{W}$  by the lottery-dependent functional:*

$$\mathbf{U}(\tilde{w}) \stackrel{\text{def}}{=} \int_a^b \mathbf{v}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}(x) dF_{\tilde{w}}(x), \quad (6)$$

where  $\mathbf{v}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}(\cdot)$  is defined according to Proposition 5 and the normalization conditions (3).

**Proof.** It is a direct consequence of the definition of regular preferences and of Proposition 6.  $\square$

Finally, note that preferences are regular if they are represented by a LS-functional and/or when marginal utility is constant.

### 3.3 Properties of disappointment models with regular preferences

Many examples of models where preferences satisfy Axioms 1 to 4 can be given. We here focus on models which are endowed with regular preferences. Before we review their properties we consider  $\mathbf{v}_z(x)$  as a function of the two variables, namely  $z$  and  $x$ , and, consequently, we set  $f(z, x) \stackrel{\text{def}}{=} \mathbf{v}_z(x)$ .

#### 3.3.1 Constant marginal utility

We first examine the case when marginal utility is constant ( $\mathbf{u}(x) = x$ ), i.e. when the functional reads:

$$\mathbf{U}(\tilde{w}) = \mathbf{c}(\tilde{w}) = \int_a^b f(\mathbb{E}[\tilde{w}], x) dF_{\tilde{w}}(x) \quad (7).$$

From the properties of  $\mathbf{v}_z(x)$  we get that  $f(z, x)$  is strictly increasing and concave with respect to  $x$  and meets the following condition:  $f(0, 0) = 0$ . It is of interest to particularize  $f(z, x)$  to get a more operational specification. This can be done through assessing additional conditions to preferences.

First, consider the risk premium  $\Pi(\tilde{w}) \stackrel{\text{def}}{=} \mathbb{E}[\tilde{w}] - \mathbf{c}(\tilde{w})$  of an arbitrary prospect  $\tilde{w} \in \mathbb{W}$ . One may assume that *risk premia are translation-invariant*, i.e.,  $\Pi(\tilde{w} + x) = \Pi(\tilde{w})$  or, equivalently,  $\mathbf{c}(\tilde{w} + y) = \mathbf{c}(\tilde{w}) + y$ . Under reasonable

mathematical assumptions, one may show that a necessary and sufficient condition for  $\Pi(\cdot)$  to exhibit the invariance property is that  $\partial f/\partial x + \partial f/\partial z = 1$ . This condition may be restated as  $f(z, x) = x + \mathcal{E}(x - z)$  where  $\mathcal{E}(\cdot)$  is strictly increasing and concave and meets the requirement:  $\mathcal{E}(0) = 0$ . Finally, the functional is that of a *disappointment model* where elation/disappointment is an increasing and convex function of the excess of the actual outcome over its expected value. It is a particular case of the model developed by L&S (1986). It may also be viewed as the opposite to a *convex measure of risk* (in the sense of Föllmer and Schied (2002)), since one may set:

$$\mathbf{r}(\tilde{w}) \stackrel{def}{=} -\mathbf{c}(\tilde{w}) = -(\mathbb{E}[\tilde{w}] + \int_a^b \mathcal{E}(x - \mathbb{E}[\tilde{w}])dF_{\tilde{w}}(x)) = -\mathbf{U}(\tilde{w}) \quad (8).$$

The interest of the above result is that it allows for *grounding a convex measure of risk on a theory of the behaviour of economic agents towards risk*. The risk controller is then assumed to behave according to Axioms 1 to 4, to exhibit constant marginal utility and to have preferences endowed with the translation invariance property.

Moreover, from Equation (8) we also get a decomposition of the risk premium  $\Pi(\tilde{w}) \stackrel{def}{=} \mathbb{E}[\tilde{w}] - \mathbf{c}(\tilde{w})$  into elementary premia, which can be viewed as the contributions of the variance, the skewness, the kurtosis ... of a random prospect to the total risk premium which is demanded by an investor. If  $\mathcal{E}(\cdot)$  is smooth enough, one may write:

$$\Pi(\tilde{w}) = -\sum_{n=2}^{+\infty} \mathbb{E}[(\tilde{w} - \mathbb{E}[\tilde{w}])^n] \mathcal{E}^{(n)}(\mathbb{E}[\tilde{w}]) / n! \quad (9).$$

The total risk premium is then an infinite sum of elementary premia, each of which is proportional to the product of two terms: the  $n$ th order centered moment of the random variable  $\tilde{w}$ , *i.e.*  $\mathbb{E}[(\tilde{w} - \mathbb{E}[\tilde{w}])^n]$ , and the  $n$ th order derivative of  $\mathcal{E}(\cdot)$  taken at point  $z = \mathbb{E}[\tilde{w}]$ . Any even moment is but a quantity of a "symmetric" risk and its coefficient must be negative if the investor is risk averse, whatever the considered definition of risk. An odd moment may be viewed as a quantity of an "asymmetric" risk and its coefficient must be positive if the investor is risk averse. Finally, Equation (9) may be viewed as a *theoretical grounding of the multimoment approach of the Capital Asset Pricing Model*. Now recall that EU theory is often violated by experiments and that no general agreement has yet been found about the explaining power of its challengers, *i.e.* Non-EU theories. Hence it is interesting to point out that, because of its flexibility, the functional (9) is compatible with many of the anomalies of financial theory.

### 3.3.2 Variable marginal utility

We now turn to the general case of variable marginal utility. The functional  $\mathbf{U}(\tilde{w})$  will now read:  $\mathbf{U}(\tilde{w}) = \mathbf{u}(\mathbf{c}(\tilde{w})) = \int_a^b f(\mathbb{E}[\mathbf{u}(\tilde{w})], x)dF_{\tilde{w}}(x)$ . Here again  $f(z, x)$  is strictly increasing and concave with respect to  $x$  and meets the following condition:  $f(0, 0) = 0$ . Since investors care but about "utils", the risk premium of an arbitrary prospect  $\tilde{w} \in \mathbb{W}$  is now defined as  $\Pi(\tilde{w}) \stackrel{def}{=} \mathbb{E}[\mathbf{u}(\tilde{w})] - \mathbf{u}(\mathbf{c}(\tilde{w}))$  and one may again assume that *risk premia are translation-invariant when they are expressed in utils, i.e. :  $\mathbf{u}(\mathbf{c}(\tilde{w}) + y) = \mathbf{u}(\mathbf{c}(\tilde{w})) + \mathbf{u}(y)$* .

Under reasonable mathematical assumptions, the functional may be identified to a LS-functional which expresses as:  $\mathbf{U}(\tilde{w}) \stackrel{def}{=} \int_a^b \mathbf{v}_{\mathbb{E}[\mathbf{u}(\tilde{w})]}(x) dF_{\tilde{w}}(x) = \mathbb{E}[\mathbf{u}(\tilde{w})] + \int_a^b \mathcal{E}(u(x) - \mathbb{E}[\mathbf{u}(\tilde{w})]) dF_{\tilde{w}}(x)$ . Elation/disappointment is an increasing and convex function of the excess of the actual outcome over its expected utility. Clearly, the above results are of interest iff  $\mathbf{u}(\cdot)$  can be elicited. This question is now going to be addressed.

### 3.3.3 The elicitation property.

We focus, in this subsection on LS-functionals, because, as it has been proved in Chauveau and Nalpas (2010), they are endowed with the elicitation property. As a preliminary, we set a new definition:

**Definition 6 (strong indifference).** *Two prospects  $\tilde{w}_1$  and  $\tilde{w}_2$  are strongly indifferent iff (a) they are indifferent and (b) they meet the betweenness property.<sup>3</sup> The binary relation " $\tilde{w}_1$  and  $\tilde{w}_2$  are strongly indifferent" will be labelled " $\tilde{w}_1 \approx \tilde{w}_2$ ".*

The binary relation  $\approx$  is obviously an equivalence relation over  $\mathbb{W}$ . Clearly, strong indifference implies indifference in the usual sense which will be called, from now on, weak indifference. The properties of LS-functionals may then be summed up in the following propositions where  $\tilde{w}_p^{a,x}$  ( $\tilde{w}_{1-q}^{y,b}$ ) denotes the binary lottery  $[a, x; 1-p, p]$  ( $[y, b; q, 1-q]$ ).

**Proposition 7 (strong indifference).** *If preferences are represented by a LS-functional, two prospects  $\tilde{w}_1$  and  $\tilde{w}_2$  are strongly indifferent iff they exhibit the same certainty equivalent and the same expected utility, what formally reads:  $\tilde{w}_1 \approx \tilde{w}_2 \Leftrightarrow \mathbf{c}(\tilde{w}_1) = \mathbf{c}(\tilde{w}_2)$  and  $\mathbb{E}[\mathbf{u}(\tilde{w}_1)] = \mathbb{E}[\mathbf{u}(\tilde{w}_2)]$*

**Proposition 8 (strong equivalents).** *If preferences are represented by a LS-functional, there exists exactly one binary lottery of the  $\tilde{w}_p^{a,x}$  type (of the  $\tilde{w}_{1-q}^{y,b}$  type) which is strongly indifferent to  $\tilde{w}$ . Lottery  $\tilde{w}_p^{a,x}$  ( $\tilde{w}_{1-q}^{y,b}$ ) will be called the left (right) strong equivalent of  $\tilde{w}$ . The degenerate lottery  $\delta(z)$  (the binary lottery  $\tilde{w}_{\mathbf{u}(z)}^{a,b}$ ) is a maximal (minimal) element in  $\mathbb{W}_{\mathbf{u}(z)}$ , i.e.  $\tilde{w}_{\mathbf{u}(z)}^{a,b} \preceq z \preceq \delta(z)$ .*

**Proofs.** The proofs are given in the Appendix.  $\square$

An important property of LS-models is the *elicitation property*. Let  $w \in [a, b]$  ( $\pi \in [0, 1]$ ) be an arbitrary level of wealth (probability). Consider the sequence of binary lotteries labelled  $\{\tilde{w}_{p_n}^{a,x_n}\}_{n \in \mathbb{N}}$  that meets the below requirements:

$$x_0 = w, p_0 = \pi \text{ and } \tilde{w}_{1-p_{n+1}}^{x_{n+1},b} \approx \tilde{w}_{p_n}^{a,x_n}$$

where  $\tilde{w}_{1-p_{n+1}}^{x_{n+1},b}$  is the right strong equivalent of  $\tilde{w}_{p_n}^{a,x_n}$ . Clearly,  $\{x_n\}_{n \in \mathbb{N}}$  is a strictly decreasing sequence. The difference between the expected utilities of two consecutive binary lotteries,  $\tilde{w}_{p_n}^{a,x_n}$  and  $\tilde{w}_{p_{n+1}}^{a,x_{n+1}}$ , is equal to the second weight  $(1-p_{n+1})$  of the right strong equivalent  $\tilde{w}_{1-p_{n+1}}^{x_{n+1},b}$  of  $\tilde{w}_{p_n}^{a,x_n}$ , what formally reads:

$$\mathbb{E}[\mathbf{u}(\tilde{w}_{p_n}^{a,x_n})] - \mathbb{E}[\mathbf{u}(\tilde{w}_{p_{n+1}}^{a,x_{n+1}})] = 1 - p_{n+1}.$$

<sup>3</sup>Recall that two prospects share the betweenness property iff for any  $\alpha \in [0, 1]$ ,  $\tilde{w}_1 \preceq \tilde{w}_2 \Rightarrow \tilde{w}_1 \preceq \alpha \tilde{w}_1 \oplus (1-\alpha) \tilde{w}_2 \preceq \tilde{w}_2$ .

Consequently, the expected utility of the initial lottery –i.e.  $\pi \mathbf{u}(w)$ – satisfies the following equality:

$$\pi \mathbf{u}(w) = \mathbb{E} [\mathbf{u}(\tilde{w}_{p_n}^{a,x_n})] + \sum_{i=1}^n (1 - p_i) \quad (10).$$

Alternatively, one may consider a sequence of binary lotteries,  $\{\tilde{w}_{q_n}^{y_n,b}\}_{n \in \mathbb{N}}$ , that are defined as indicated below:

$$y_0 = w; q_0 = \pi \quad \text{and} \quad \tilde{w}_{q_{n+1}}^{a,y_{n+1}} \approx \tilde{w}_{1-q_n}^{y_n,b}$$

and the elements of the sequence are endowed with the following property:

$$\mathbb{E} [\mathbf{u}(\tilde{w}_{q_{n+1}}^{a,y_{n+1}})] - \mathbb{E} [\mathbf{u}(\tilde{w}_{q_n}^{a,y_n})] = 1 - q_{n+1}, \text{ or, equivalently,}$$

$$\pi \mathbf{u}(w) = \mathbb{E} [\mathbf{u}(\tilde{w}_{q_n}^{a,y_n})] - \sum_{i=1}^n (1 - q_i) \quad (11).$$

From now on, the sequences  $\{\tilde{w}_{p_n}^{a,x_n}\}_{n \in \mathbb{N}}$  and  $\{\tilde{w}_{1-q_n}^{y_n,b}\}_{n \in \mathbb{N}}$ , will be called the canonical sequences generated by  $(w, \pi)$ . As shown below, they respectively converge, in LS-models, towards  $\delta(a)$  or  $\delta(b)$ . The result holds whatever the value of  $\pi$ .

**Proposition 9 (elicitation property).** *Let  $\{\tilde{w}_{p_n}^{a,x_n}\}_{n \in \mathbb{N}}$  and  $\{\tilde{w}_{q_n}^{y_n,b}\}_{n \in \mathbb{N}}$  be the canonical sequences of binary lotteries generated by  $(w, \pi) \in ]a, b[ \times ]0, 1[$ . Assume that preferences are represented by a LS-functional and that investors are disappointment averse. Then,  $\{x_n\}_{n \in \mathbb{N}}$  ( $\{y_n\}_{n \in \mathbb{N}}$ ) is a decreasing (increasing) sequence of real numbers converging towards  $a$  ( $b$ ). The sequence  $\{1 - p_n\}_{n \in \mathbb{N}}$  ( $\{1 - q_n\}_{n \in \mathbb{N}}$ ) is increasing and converges towards  $\pi \ell$  ( $1 - \pi \ell$ ) where  $\ell$  does not depend on  $\pi$  and is a strictly increasing function of  $w$ , mapping  $[a, b]$  on to  $[0, 1]$ .*

**Proof.** It is given in the Appendix.  $\square$

Finally, in LS-models, we have the following equalities:

$$l = \lim_{n \rightarrow \infty} (\sum_{i=1}^n (1 - p_i)) / \pi = \lim_{n \rightarrow \infty} (1 - \sum_{i=0}^n (1 - q_i)) / \pi$$

and from now on we shall set  $l = \mathbf{u}(w)$ . Finally, note that Axioms 1 to 4 are, at least in principle, experimentally testable since their checking comes down to making choices between binary loteries. The number of experiments obviously depends on the desired accuracy of the elementary utility function  $\mathbf{u}(\cdot)$ . The last remark is that EU theory is clearly a degenerate case of the above disappointment theory.

## 4 Concluding remarks

In this paper, a fully choice-based theory of disappointment has been developed which can be viewed as an axiomatic foundation of models *à la* L&S (1986). The above theory of disappointment may be generalized in three ways: the first consists in relaxing the assumption of a concave or convex *n.e.u.* function. One can also weaken the assumption of functions which are continuously derivable and allow for discontinuities on a finite set of values of their argument. Last, the assumption made at the beginning of Section 3, according to which no violations of SSD occur, is convenient but not essential.

## References

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## 5 Appendix (Proofs)

*Proof of Proposition 1.*

Let  $\tilde{\omega}_i = u(\tilde{w}_i)$  for  $i = 1, 2$ . By definition of *SSSD*, it is equivalent to state

(a)  $\tilde{w}_2 \succsim_2^u \tilde{w}_1$ ,

(b)  $\tilde{\omega}_2 \succsim_2 \tilde{\omega}_1$ , or

(c)  $\int_0^v [F_{\tilde{\omega}_1}(t) - F_{\tilde{\omega}_2}(t)] dt \leq 0$  for  $v \in [0, 1]$ . This last condition is, in its turn equivalent to the following one:

$\int_a^z u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \leq 0$  for any  $z \in [a, b]$ , because of the following equality:

$$\int_0^v [F_{\tilde{\omega}_1}(t) - F_{\tilde{\omega}_2}(t)] dt = \int_a^{u^{-1}(v)} (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) u'(x) dx. \quad \square$$

*Proof of Proposition 2.*

We want to prove that if  $\tilde{w}_1$  dominates  $\tilde{w}_2$  by *SSSD*, then  $\tilde{w}_1$  is preferred to  $\tilde{w}_2$  or, equivalently, that  $\mathbf{U}(\tilde{w}_1) - \mathbf{U}(\tilde{w}_2) \geq 0$ . To do so, consider two prospects  $\tilde{w}_1$  and  $\tilde{w}_2$ . Let

$$\int_a^b u'(t) F_{\tilde{w}_i}(t) dt = - \int_a^b u(x) dF_{\tilde{w}_i} = -\mathbb{E}[\tilde{w}_i] = -\lambda_i$$

for  $i = 1, 2$ . Assume that  $\tilde{w}_1$  dominates  $\tilde{w}_2$  by *SSSD*, we get:

$$\int_a^b u'(t) (F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t)) dt = -(\lambda_1 - \lambda_2) \leq 0 \Rightarrow \lambda_1 - \lambda_2 \geq 0$$

Next, since  $\mathcal{E}(\cdot)$  is strictly increasing we also get:

$$\mathcal{E}(u(x) - \lambda_1) \leq \mathcal{E}(u(x) - \lambda_2)$$

and the difference between the two functionals expresses as:

$$\begin{aligned}\mathbf{U}(\tilde{w}_1) - \mathbf{U}(\tilde{w}_2) &= \int_a^b (u(x) + \mathcal{E}(u(x) - \lambda_1)) dF_{\tilde{w}_1}(x) \\ &\quad - \int_a^b (u(x) + \mathcal{E}(u(x) - \lambda_2)) dF_{\tilde{w}_2}(x) \\ &= (\lambda_1 - \lambda_2) \\ &\quad + \int_a^b \mathcal{E}(u(x) - \lambda_1) dF_{\tilde{w}_1}(x) - \int_a^b \mathcal{E}(u(x) - \lambda_2) dF_{\tilde{w}_2}(x)\end{aligned}$$

and, finally:

$$\mathbf{U}(\tilde{w}_1) - \mathbf{U}(\tilde{w}_2) = T1 + T2$$

where:

$$T1 = (\lambda_1 - \lambda_2) + \int_a^b (\mathcal{E}(u(x) - \lambda_1) - \mathcal{E}(u(x) - \lambda_2)) dF_{\tilde{w}_2}(x)$$

and:

$$T2 = \int_a^b \mathcal{E}(u(x) - \lambda_1) (dF_{\tilde{w}_1}(x) - dF_{\tilde{w}_2}(x))$$

Straightforward calculations give:

$$T1 = (\lambda_1 - \lambda_2) - \int_a^b (\lambda_1 - \lambda_2) \mathcal{E}'(u(x) - \lambda_1 + \theta_1(\lambda_1 - \lambda_2)) dF_{\tilde{w}_2}(x)$$

with  $\theta_1 \in [0, 1]$  and:

$$\begin{aligned}T2 &= [\mathcal{E}(u(x) - \lambda_1) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))]_a^b \\ &\quad - \int_a^b \mathcal{E}'(u(x) - \lambda_1) u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \\ &= \mathcal{E}(-\lambda_1) (F_{\tilde{w}_1}(a) - F_{\tilde{w}_2}(a)) - \int_a^b \mathcal{E}'(u(x) - \lambda_1) u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \\ &= \mathcal{E}(-\lambda_1) (F_{\tilde{w}_1}(a) - F_{\tilde{w}_2}(a)) - \mathcal{E}'(1 - \lambda_1) \int_a^b u'(t) (F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t)) dt \\ &\quad + \int_a^b \mathcal{E}''(u(x) - \lambda_1) u'(x) \left[ \int_a^x u'(t) (F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t)) dt \right] dx\end{aligned}$$

Clearly, the following inequality:  $[1 - \sup \mathcal{E}'(z)] \geq 0$  implies the following equality:  $\text{sign}(T1) = \text{sign}(\lambda_1 - \lambda_2)$ . Hence,  $T1$  is positive since  $\lambda_1 - \lambda_2 \geq 0$ . The term  $T2$  is also positive, since it is the sum of three positive terms:

(a)  $\mathcal{E}(-\lambda_1) (F_{\tilde{w}_1}(a) - F_{\tilde{w}_2}(a))$  is positive because  $\mathcal{E}(-\lambda_1)$  is negative and so is  $(F_{\tilde{w}_1}(a) - F_{\tilde{w}_2}(a))$  (from *SSSD*)

(b) the second term is positive because  $\mathcal{E}'(1 - \lambda_1)$  is positive, and the integral  $\int_a^b u'(t) (F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t)) dt$  is negative (from *SSSD*).

(c) the last term is positive because  $\mathcal{E}''(u(x) - \lambda_1)$  is negative,  $u'(x)$  is positive and  $\int_a^x u'(t)(F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t))dt$  is negative (from *SSSD*). Finally,  $\mathbf{U}(\tilde{w}_1) - \mathbf{U}(\tilde{w}_2) \geq 0 \square$

*Proof of Proposition 3.*

As a preliminary, recall that  $u(\cdot)$  is more concave than  $v(\cdot)$  if and only if  $u \circ v^{-1}(\cdot)$  is concave *i.e.* if there exists  $g(\cdot)$  mapping  $[0, 1]$  on to itself and such that:  $u(x) = g \circ v(x)$  with  $g'(\cdot) > 0$  and  $g''(\cdot) < 0$

The proof is grounded on the following calculations:

$$\begin{aligned} \int_a^z u'(x)(F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))dx &= \int_a^z g'(v(x))v'(x)(F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))dx \\ &= \left[ g'(v(x)) \int_a^x v'(t)(F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t))dt \right]_a^z \\ &\quad - \int_a^z g''(v(x))v'(x) \left[ \int_a^x v'(t)(F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t))dt \right] dx \\ &= g'(v(z)) \int_a^z v'(t)(F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t))dt \\ &\quad - \int_a^z g''(v(x))v'(x) \left[ \int_a^x v'(t)(F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t))dt \right] dx \end{aligned}$$

Finally, we get the following equivalences and/or implications:

$$\int_a^z v'(t)(F_{\tilde{w}_1}(t) - F_{\tilde{w}_2}(t))dt < 0 \text{ for any } z \Rightarrow \int_a^z u'(x)(F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))dx < 0 \text{ for any } z$$

or:

$$\tilde{w}_1 \preceq_2^v \tilde{w}_2 \Rightarrow \tilde{w}_1 \preceq_2^u \tilde{w}_2 \quad \Leftrightarrow \quad \mathbf{W}_2^v \subset \mathbf{W}_2^u$$

and, as a consequence:

$$\mathbf{W}_2^{v-} \subseteq \mathbf{W}_2^{u-} \text{ and } \mathbf{W}_2^{v+} \subseteq \mathbf{W}_2^{u+}$$

$\square$

*Proof of Proposition 4.*

Two cases may occur: either *SSD* is violated or not.

**A.** We first assume that *SSD* is not violated. As a consequence, there exists at least one concave function which is consistent. It is the *n.e.u.* affine function defined by  $\mathbf{f}(x) = (x - a) / (b - a)$ . Hence,  $\overline{\mathbf{U}}_C \neq \emptyset$ .

The subset  $\overline{\mathbf{U}}_C$  may then include one element or more. In the first case  $\overline{\mathbf{U}}_C = \{\mathbf{f}(\cdot)\}$  and Proposition 4 is clearly valid. We now leave aside this trivial case and assume that  $\overline{\mathbf{U}}_C$  includes at least two elements, *i.e.* at least one strictly concave *n.e.u.* function.

Consider two *n.e.u.* functions,  $u(\cdot)$  and  $v(\cdot)$ . Then, from Proposition 3, we get the following results:

(a) if  $u(\cdot)$  is consistent and is more concave than  $v(\cdot)$ , then  $v(\cdot)$  is also consistent.

(b) if  $u(\cdot)$  and  $v(\cdot)$  are concave and consistent, then  $\text{Min}(u, v)$  is consistent.

(c) Similarly, If  $v(\cdot)$  is inconsistent and less concave than  $u(\cdot)$  then  $u(\cdot)$  is also inconsistent.

(d) if  $u(\cdot)$  and  $v(\cdot)$  are concave and if  $\text{Min}(u, v)$  is inconsistent, then  $u(\cdot)$  and  $v(\cdot)$  are inconsistent.

(e) if  $u(\cdot)$  is more concave than  $v(\cdot)$  then  $v(\cdot) \in \text{hypo}(\mathbf{u})$ , where  $\text{hypo}(\mathbf{u})$  is the hypograph of  $u$ .

Let  $\mathbf{u}(\cdot)$  denote the lower envelope of the functions belonging to  $\overline{\mathbb{U}}_C$ . We want to prove that  $\mathbf{W}_2^{\mathbf{u}-} = \emptyset$ . The proof is three-step.

The first step consists in defining a consistent concave n.e.u. function  $u(\cdot)$  which is close to  $\mathbf{u}(\cdot)$ . First, recall that the hypograph of  $\mathbf{u}(\cdot)$  is defined as  $\text{hypo}(\mathbf{u}) = \bigcap_{u \in \overline{\mathbb{U}}_C} \text{hypo}(u)$  and, consequently, any concave n.e.u. function whose hypograph is strictly included in  $\text{hypo}(\mathbf{u})^4$  is consistent.

Now let  $u(\cdot)$  be defined by the following equality:

$$u(x) \stackrel{\text{def}}{=} \mathbf{u}(x) - y(x)$$

where

$$y(x) = \eta \left( \frac{x-a}{b-a} \right) - \eta \left( \frac{x-a}{b-a} \right)^2$$

Clearly,  $y(x) \geq 0$  for  $x \in [a, b]$ ,  $y'(x) \geq 0$  for  $x \in [a, a + (b-a)/2]$ ,  $y'(x) \leq 0$  for  $x \in [a + (b-a)/2, b]$ ,  $y(a) = y(b) = 0$ ,  $a + (b-a)/2 = \text{Arg max}[y(x)]$  and  $\max[y(x)] = \eta/4$ . A sufficient condition for  $u(\cdot)$  to be concave is that:

$$\eta < \frac{1}{2} (b-a)^2 \inf_{x \in [a, b]} (-\mathbf{u}''(x))$$

Moreover,  $u(\cdot)$  will be strictly increasing if  $u'(x)$  is strictly positive. A sufficient condition for this is that:

$$\eta < (b-a) \inf_{x \in [a, b]} (\mathbf{u}'(x))$$

and, finally,  $u(\cdot)$  is concave and strictly increasing if the real number  $\eta$  satisfies the below inequality:

$$\eta < \min \left[ \frac{1}{2} (b-a)^2 \inf_{x \in [a, b]} (-\mathbf{u}''(x)), (b-a) \inf_{x \in [a, b]} (\mathbf{u}'(x)) \right] \quad (1)$$

Since  $u(a) = \mathbf{u}(a) = 0$  and  $u(b) = \mathbf{u}(b) = 1$ ,  $u(\cdot)$  is normalized and since  $\eta > 0$ , the hypograph of  $\mathbf{u}(\cdot)$  strictly includes that of  $u(\cdot)$  and  $u(\cdot)$  is consistent. Finally, the function  $u(\cdot)$  is a concave n.e.u. function if (1) is met.

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<sup>4</sup> i.e. there exists at least one value of  $x \in [a, b]$  such that  $u(x) < \mathbf{u}(x)$ .

The second step consists in looking for an upper bound for the following difference:

$$\Delta = \left| \int_a^z \mathbf{u}'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx - \int_a^z u'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \right|$$

Integrating by parts, we get:

$$\begin{aligned} \Delta &= \left| \int_a^z (\mathbf{u}'(x) - u'(x)) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \right| \\ &= \left| (\mathbf{u}(z) - u(z)) (F_{\tilde{w}_1}(z) - F_{\tilde{w}_2}(z)) + \int_a^z (\mathbf{u}'(x) - u'(x)) (dF_{\tilde{w}_1}(x) - dF_{\tilde{w}_2}(x)) \right| \end{aligned}$$

and, consequently:

$$\Delta \leq |(\mathbf{u}(z) - u(z)) (F_{\tilde{w}_1}(z) - F_{\tilde{w}_2}(z))| + \left| \int_a^z (\mathbf{u}'(x) - u'(x)) (dF_{\tilde{w}_1}(x) - dF_{\tilde{w}_2}(x)) \right| \quad (2)$$

The first term is bounded indicated as below:

$$|(\mathbf{u}(z) - u(z)) (F_{\tilde{w}_1}(z) - F_{\tilde{w}_2}(z))| \leq |(\mathbf{u}(z) - u(z))| \leq \sup_{z \in [a, b]} |\mathbf{u}'(z) - u'(z)|$$

We now show that the second term may be bounded as indicated below

$$\left| \int_a^z (\mathbf{u}'(x) - u'(x)) (dF_{\tilde{w}_1}(x) - dF_{\tilde{w}_2}(x)) \right| \leq 2 \sup_{z \in [a, b]} |\mathbf{u}'(z) - u'(z)|$$

Indeed, we have

$$\left| \int_a^z (\mathbf{u}'(x) - u'(x)) (dF_{\tilde{w}_1}(x) - dF_{\tilde{w}_2}(x)) \right| \leq \left| \int_a^z (\mathbf{u}'(x) - u'(x)) dF_{\tilde{w}_1}(x) \right| + \left| \int_a^z (\mathbf{u}'(x) - u'(x)) dF_{\tilde{w}_2}(x) \right|$$

and, for  $i = 1, 2$ :

$$\left| \int_a^z (\mathbf{u}'(x) - u'(x)) dF_{\tilde{w}_i}(x) \right| \leq \sup_{z \in [a, b]} |\mathbf{u}'(z) - u'(z)| \int_a^z dF_{\tilde{w}_i}(x) \leq \sup_{z \in [a, b]} |\mathbf{u}'(z) - u'(z)|$$

Finally, an upper bound of  $\Delta$  is given by the following inequality:

$$\Delta \leq 3 \sup_{z \in [a, b]} |\mathbf{u}'(z) - u'(z)|$$

$$\text{Now, recall that } \sup_{z \in [a, b]} |\mathbf{u}'(z) - u'(z)| = \sup_{z \in [a, b]} \left| \eta \left( \frac{z-a}{b-a} \right) - \eta \left( \frac{z-a}{b-a} \right)^2 \right| = \eta/4.$$

As a consequence, we get

$$\Delta \leq 3\eta/4$$

The last step consists in showing that if  $\mathbf{u}(\cdot)$  were not consistent, then we would get a contradiction. Indeed if  $\mathbf{u}(\cdot)$  were not consistent there would exist two prospects  $\tilde{w}_1$  and  $\tilde{w}_2$  such that  $\tilde{w}_1 \preceq \tilde{w}_2$  and, simultaneously, there would exist  $z \in [a, b]$ , such that  $\int_a^z \mathbf{u}'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx > 0$ . In other words, there would exist a strictly positive real number  $\epsilon$  such that

$$\int_a^z \mathbf{u}'(x) (F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x)) dx \geq \epsilon > 0$$

Actually, since  $u(\cdot)$  is consistent, we must have  $\int_a^z u'(x)(F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))dx < 0$  and, consequently: we get:

$$\Delta = \int_a^z \mathbf{u}'(x)(F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))dx + \left| \int_a^z u'(x)(F_{\tilde{w}_1}(x) - F_{\tilde{w}_2}(x))dx \right| \geq \epsilon$$

and, finally:

$$\epsilon \leq 3\eta/4$$

Hence, if  $\eta$  is small enough, *i.e.* if  $\eta < 4\epsilon/3$ , we get a contradiction and, finally,  $\mathbf{u}(\cdot)$  is well consistent.

**B.** We now assume that *SSD* is not violated, that is when  $\mathbf{W}_2^{\mathbf{f}-} \neq \emptyset$ . Then, no concave *n.e.u.* functions may be consistent. Indeed, if  $u$  were a consistent and concave *n.e.u.* function, then, from Proposition 3,  $\mathbf{f}(\cdot)$  would be consistent, which would contradict the initial assumption. By contrast, the subset of convex *n.e.u.* functions is never empty since it always includes the following function:  $\underline{u}(x) = 0$  for  $x \in [a, b[$  and  $\underline{u}(b) = 1$ . The rest of the proof is analogous to the above one.  $\square$

*Proof of Proposition 7.*

The first part of the proof consists in proving that, in LS-models, two indifferent prospects  $\tilde{w}_1$  and  $\tilde{w}_2$  which have the same expected utility  $\bar{\mathbf{u}}$  and the same certainty equivalent  $\mathbf{c}$ , are strongly indifferent. Let  $\tilde{w}_1$  and  $\tilde{w}_2$  exhibit the same expected utility  $\bar{\mathbf{u}}$  and the same certainty equivalent  $\mathbf{c}$ . From (2) we get, for  $i = 1, 2$ :

$$\mathbf{u}(\mathbf{c}) = \bar{\mathbf{u}} + \sum_{n=1}^N p_n^i (\mathcal{E}(\mathbf{u}(w_n)) - \bar{\mathbf{u}})$$

where  $\tilde{w}_i = [w_1, \dots, w_N; p_1^i, \dots, p_N^i]$  ( $i = 1, 2$ ) and where  $\bar{\mathbf{u}} = \sum_{n=1}^N p_n^i \mathbf{u}(w_n)$ . As a consequence, we have:

$$\sum_{n=1}^N p_n^1 \mathcal{E}(\mathbf{u}(w_n)) - \bar{\mathbf{u}} - \sum_{n=1}^N p_n^2 \mathcal{E}(\mathbf{u}(w_n)) - \bar{\mathbf{u}} = 0 \quad (3)$$

Now, consider the compound lottery

$$\tilde{w}_\alpha \stackrel{def}{=} \alpha \tilde{w}_1 \oplus (1 - \alpha) \tilde{w}_2 = [w_1, \dots, w_N; \alpha p_1^1 + (1 - \alpha) p_1^2, \dots, \alpha p_N^1 + (1 - \alpha) p_N^2]$$

Its expected utility is:

$$\mathbb{E}[\mathbf{u}(\tilde{w}_\alpha)] = \sum_{n=1}^N (\alpha p_n^1 + (1 - \alpha) p_n^2) \mathbf{u}(w_n) = \bar{\mathbf{u}}$$

From (??) we get:

$$\mathbf{u}(\mathbf{c}(\tilde{w}_\alpha)) = \bar{\mathbf{u}} + \sum_{n=1}^N (\alpha p_n^1 + (1 - \alpha) p_n^2) (\mathcal{E}(\mathbf{u}(w_n)) - \bar{\mathbf{u}})$$

where  $\mathbf{c}(\tilde{w}_\alpha)$  is the certainty equivalent of  $\tilde{w}_\alpha$  and, finally:

$$\mathbf{u}(\mathbf{c}(\tilde{w}_\alpha)) - \mathbf{u}(\mathbf{c}) = \alpha \left( \sum_{n=1}^N p_n^1 \mathcal{E}(\mathbf{u}(w_n)) - \bar{\mathbf{u}} - \sum_{n=1}^N p_n^2 \mathcal{E}(\mathbf{u}(w_n)) - \bar{\mathbf{u}} \right) = 0$$

The proof of the converse is as follows. Consider two discrete prospects:

$$\tilde{w}_i = [w_1, \dots, w_N ; p_1^i, \dots, p_N^i] \quad i = 1, 2$$

and their probability mixture:

$$\alpha \tilde{w}_1 \oplus (1 - \alpha) \tilde{w}_2 = [w_1, \dots, w_N ; \alpha p_1^1 + (1 - \alpha) p_1^2, \dots, \alpha p_N^1 + (1 - \alpha) p_N^2]$$

where  $\alpha \in [0, 1]$ . We must show that if  $\tilde{w}_1$  and  $\tilde{w}_2$  are strongly indifferent *-i.e.* if they have the same certainty equivalent and if they exhibit the betweenness property-, then they exhibit the same expected utility. We have, for  $i = 1, 2$ :

$$\mathbf{u}(c) = \mathbf{u}(\mathbf{c}(\tilde{w}_i)) = \bar{\mathbf{u}}_i + \sum_{n=1}^N p_n^i \mathcal{E}(\mathbf{u}_n^i)$$

where:

$$\bar{\mathbf{u}}_i = \sum_{n=1}^N p_n^i \mathbf{u}(w_n) \quad \text{and} \quad \mathbf{u}_n^i = \mathbf{u}(w_n) - \bar{\mathbf{u}}_i \quad (4)$$

By definition, we have:

$$\mathbf{u}(\mathbf{c}(\alpha \tilde{w}_1 \oplus (1 - \alpha) \tilde{w}_2)) = \alpha \bar{\mathbf{u}}_1 + (1 - \alpha) \bar{\mathbf{u}}_2 + \sum_{n=1}^N [\alpha p_n^1 + (1 - \alpha) p_n^2] \mathcal{E}(\alpha \mathbf{u}_n^1 + (1 - \alpha) \mathbf{u}_n^2)$$

Now, from (??), we get:

$$\alpha \mathbf{u}(\mathbf{c}(\tilde{w}_1)) + (1 - \alpha) \mathbf{u}(\mathbf{c}(\tilde{w}_2)) = \alpha \bar{\mathbf{u}}_1 + (1 - \alpha) \bar{\mathbf{u}}_2 + \sum_{n=1}^N \alpha p_n^1 \mathcal{E}(\mathbf{u}_n^1) + \sum_{n=1}^N (1 - \alpha) p_n^2 \mathcal{E}(\mathbf{u}_n^2)$$

Subtracting the above equation from the previous one and using the betweenness property yields:

$$\begin{aligned} \sum_{n=1}^N p_n^1 \alpha \mathcal{E}(\mathbf{u}_n^1) + \sum_{n=1}^N p_n^2 (1 - \alpha) \mathcal{E}(\mathbf{u}_n^2) &= \alpha \left( \sum_{n=1}^N p_n^1 \mathcal{E}(\alpha \mathbf{u}_n^1 + (1 - \alpha) \mathbf{u}_n^2) \right) \\ &\quad + (1 - \alpha) \left( \sum_{n=1}^N p_n^2 \mathcal{E}(\alpha \mathbf{u}_n^1 + (1 - \alpha) \mathbf{u}_n^2) \right) \end{aligned}$$

or, equivalently:

$$\begin{aligned} &\sum_{n=1}^N p_n^1 \alpha \mathcal{E}(\mathbf{u}_n^1) + \sum_{n=1}^N p_n^2 (1 - \alpha) \mathcal{E}(\mathbf{u}_n^1) + \sum_{n=1}^N p_n^2 (1 - \alpha) (\mathcal{E}(\mathbf{u}_n^2) - \mathcal{E}(\mathbf{u}_n^1)) \\ &= \sum_{n=1}^N [\alpha p_n^1 + (1 - \alpha) p_n^2] \mathcal{E}(\alpha \mathbf{u}_n^1 + (1 - \alpha) \mathbf{u}_n^2) \end{aligned}$$

and, finally:

$$\begin{aligned} &\sum_{n=1}^N [\alpha p_n^1 + (1 - \alpha) p_n^2] \left[ \frac{\mathcal{E}(\mathbf{u}_n^1) - \mathcal{E}(\alpha \mathbf{u}_n^1 + (1 - \alpha) \mathbf{u}_n^2)}{\mathcal{E}(\mathbf{u}_n^1) - \mathcal{E}(\mathbf{u}_n^2)} \right] = \sum_{n=1}^N p_n^2 (1 - \alpha) (\mathcal{E}(\mathbf{u}_n^1) - \mathcal{E}(\mathbf{u}_n^2)) \\ &\sum_{n=1}^N [\alpha p_n^1 + (1 - \alpha) p_n^2] \left[ \frac{\mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_1) - \mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_2)}{-\mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_1) + \mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_2)} \right] (1 - \alpha)^{-1} = \sum_{n=1}^N p_n^2 \left( \frac{\mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_1) - \mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_2)}{-\mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_1) + \mathcal{E}(\mathbf{u}(w_n) - \bar{\mathbf{u}}_2)} \right) \\ &\sum_{n=1}^N \varpi_n(\alpha) (\mathbf{u}_n^1 - \mathbf{u}_n^2) \mathcal{E}' \left( \frac{\mathbf{u}(w_n) - \bar{\mathbf{u}}_1}{+\theta_n(\alpha) (\mathbf{u}_n^1 - \mathbf{u}_n^2)} \right) = \sum_{n=1}^N p_n^2 (\mathbf{u}_n^1 - \mathbf{u}_n^2) \\ &\quad \times \mathcal{E}' \left( \frac{\mathbf{u}(w_n) - \bar{\mathbf{u}}_1}{+\zeta_n(\mathbf{u}_n^1 - \mathbf{u}_n^2)} \right) \\ &(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2) \left\{ \sum_{n=1}^N \varpi_n(\alpha) \mathcal{E}' \left( \frac{\mathbf{u}(w_n) - \bar{\mathbf{u}}_1}{+\theta_n(\alpha) (\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2)} \right) \right\} = (\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2) \\ &\quad \times \left\{ \sum_{n=1}^N p_n^2 \mathcal{E}' \left( \frac{\mathbf{u}(w_n) - \bar{\mathbf{u}}_1}{+\zeta_n(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2)} \right) \right\} \\ &(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2) F(\alpha) = (\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2) \Lambda \end{aligned}$$

Since  $F(\alpha)$  cannot be equal to  $\Lambda$  for any value of  $\alpha$ , we must have  $\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2 = 0$ .  $\square$

*Proof of Proposition 8.*

Let:

$$\tilde{w}_{\pi}^{a,b} \stackrel{def}{=} [a, b; 1 - \pi, \pi], \quad \tilde{w}_{z,\pi}^{\alpha} \stackrel{def}{=} \alpha \delta(z) \oplus (1 - \alpha) \tilde{w}_{\pi}^{a,b}$$

and:

$$\bar{\mathbf{u}} \stackrel{def}{=} \alpha \mathbf{u}(z) + (1 - \alpha) \pi = \mathbb{E}[\mathbf{u}(\tilde{w}_{\pi}^{\alpha})]$$

In LS-models we get:

$$\mathbf{U}(\tilde{w}_{z,\pi}^{\alpha}) \stackrel{def}{=} \bar{\mathbf{u}} + \alpha \mathcal{E}(\mathbf{u}(z) - \bar{\mathbf{u}}) + (1 - \alpha) \pi \mathcal{E}(1 - \bar{\mathbf{u}}) + (1 - \alpha) (1 - \pi) \mathcal{E}(-\bar{\mathbf{u}})$$

and, consequently:

$$\mathbf{U}(\tilde{w}_{z,\pi}^{\alpha}) = \alpha \mathbf{U}(\delta(z)) + (1 - \alpha) \mathbf{U}(\tilde{w}_{\pi}^{a,b}) + EXP$$

where:

$$EXP \stackrel{def}{=} \alpha \mathcal{E}(\mathbf{u}(z) - \bar{\mathbf{u}}) + (1 - \alpha) \left\{ \begin{array}{l} \pi (\mathcal{E}(1 - \bar{\mathbf{u}}) - \mathcal{E}(1 - \pi)) \\ + (1 - \pi) (\mathcal{E}(-\bar{\mathbf{u}}) - \mathcal{E}(-\pi)) \end{array} \right\}$$

We get:

$$\begin{aligned} EXP &= \alpha (\mathbf{u}(z) - \bar{\mathbf{u}}) \mathcal{E}'(\theta(\mathbf{u}(z) - \bar{\mathbf{u}})) + (1 - \alpha) \pi (\pi - \bar{\mathbf{u}}) \mathcal{E}'(1 - \pi + \zeta(\pi - \bar{\mathbf{u}})) \\ &\quad + (1 - \alpha) (1 - \pi) (\pi - \bar{\mathbf{u}}) \mathcal{E}'(-\pi + \xi(\pi - \bar{\mathbf{u}})) \end{aligned}$$

or:

$$\begin{aligned} EXP &= \alpha (1 - \alpha) (\mathbf{u}(z) - \pi) \mathcal{E}'(\theta(1 - \alpha) (\mathbf{u}(z) - \pi)) \\ &\quad + (1 - \alpha) \alpha (\pi - \mathbf{u}(z)) \left\{ \begin{array}{l} \pi \mathcal{E}'(1 - \pi + \zeta \alpha (\pi - \mathbf{u}(z))) \\ + (1 - \pi) \mathcal{E}'(-\pi + \xi \alpha (\pi - \mathbf{u}(z))) \end{array} \right\} \end{aligned}$$

and, finally:

$$EXP = \alpha (1 - \alpha) (\mathbf{u}(z) - \pi) \left[ - \left\{ \begin{array}{l} \mathcal{E}'(\theta(1 - \alpha) (\mathbf{u}(z) - \pi)) \\ \pi \mathcal{E}'(1 - \pi + \zeta \alpha (\pi - \mathbf{u}(z))) \\ + (1 - \pi) \mathcal{E}'(-\pi + \xi \alpha (\pi - \mathbf{u}(z))) \end{array} \right\} \right]$$

The above condition can be rewritten as:

$$EXP = \alpha (1 - \alpha) (\mathbf{u}(z) - \pi) \{ \mathcal{E}'(\theta(1 - \alpha) (\mathbf{u}(z) - \pi)) - exp \}$$

where:

$$\begin{aligned} exp &= \pi \mathcal{E}'(1 - \pi + \zeta \alpha (\pi - \mathbf{u}(z))) + (1 - \pi) \mathcal{E}'(-\pi + \xi \alpha (\pi - \mathbf{u}(z))) \\ &= \pi \mathcal{E}'(0) + \pi \zeta \alpha (\pi - \mathbf{u}(z)) \mathcal{E}''(\zeta s \alpha (\pi - \mathbf{u}(z)) + (1 - \pi) \mathcal{E}'(0) \\ &\quad + (1 - \pi) \zeta \alpha (\pi - \mathbf{u}(z)) \mathcal{E}''(\xi k \alpha (\pi - \mathbf{u}(z))) \end{aligned}$$

Note that we have:

$$\begin{aligned}\mathcal{E}'(\theta(1-\alpha)(\mathbf{u}(z)-\pi)) - exp &= \mathcal{E}'(0) - \{\pi\mathcal{E}'(0) + (1-\pi)\mathcal{E}'(0)\} \\ &+ \theta(1-\alpha)(\mathbf{u}(z)-\pi)\mathcal{E}''(\theta t(1-\alpha)(\mathbf{u}(z)-\pi)) \\ &- \pi\zeta\alpha(\pi-\mathbf{u}(z))\mathcal{E}''(\zeta s\alpha(\pi-\mathbf{u}(z))) \\ &- (1-\pi)\zeta\alpha(\pi-\mathbf{u}(z))\mathcal{E}''(\xi k\alpha(\pi-\mathbf{u}(z)))\end{aligned}$$

and, finally:

$$\begin{aligned}EXP &= \alpha(1-\alpha)(\mathbf{u}(z)-\pi)^2 \left\{ \begin{aligned} &\theta(1-\alpha)\mathcal{E}''(\theta t(1-\alpha)(\mathbf{u}(z)-\pi)) \\ &+ \pi\zeta\alpha\mathcal{E}''(\zeta s\alpha(\pi-\mathbf{u}(z))) \\ &+ (1-\pi)\zeta\alpha\mathcal{E}''(\xi k\alpha(\pi-\mathbf{u}(z))) \end{aligned} \right\} < 0 \\ &\simeq \alpha(1-\alpha)(\mathbf{u}(z)-\pi)^2\mathcal{E}''(0)\{\theta(1-\alpha)+\zeta\alpha\}\end{aligned}$$

and the condition  $EXP = 0$  implies  $\mathbf{u}(z) - \pi = 0$ .  $\square$

*Proof of Proposition 9.*

If  $x_{n+1}$  were greater than  $x_n$ ,  $\tilde{w}_{1-p_{n+1}}^{x_{n+1},b}$  would exhibit first-order stochastic dominance over  $\tilde{w}_{p_n}^{a,x_n}$ . Hence,  $x_{n+1}$  is lower than  $x_n$  and  $\{x_n\}_{n \in \mathbb{N}}$  is a decreasing sequence. It is also bounded below by  $a$ . Consequently, it converges towards a limit  $\ell \geq a$ . Next, note that the two strongly indifferent lotteries  $\tilde{w}_{p_n}^{a,x_n}$  and  $\tilde{w}_{1-p_{n+1}}^{x_{n+1},b}$  have the same expected utility, *i.e.*, we have:

$$p_n u(x_n) = p_{n+1} u(x_{n+1}) + (1-p_{n+1}) \quad \text{for } n = 0, 1, \dots \quad (5)$$

and summing the members of the above equalities yields:

$$\pi u(w) = p_n u(x_n) + \sum_{i=1}^n (1-p_i) \quad \text{for } n = 1, 2, \dots$$

The above equality implies  $S_n \stackrel{def}{=} \sum_{i=1}^n (1-p_i) \leq \pi u(w)$ . Since  $\{S_n\}_{n \in \mathbb{N}^*}$  is an increasing sequence, it converges towards a limit  $\Sigma \leq \pi u(w)$ . As a consequence,  $S_n - S_{n-1} = (1-p_n) \rightarrow 0$ , *i.e.*  $p_n \rightarrow 1$ . Moreover, since we have:  $\tilde{w}_{p_{n+1}}^{a,x_{n+1}} \prec \tilde{w}_{1-p_{n+1}}^{x_{n+1},b} \sim \tilde{w}_{p_n}^{a,x_n}$ , the sequence of binary lotteries  $\{\tilde{w}_{p_n}^{a,x_n}\}_{n \in \mathbb{N}}$  is decreasing and converges towards  $\tilde{w}_1^{a,l} = \delta(l)$ . Similarly,  $\{\tilde{w}_{1-p_n}^{x_n,b}\}_{n \in \mathbb{N}^*}$  converges towards  $\tilde{w}_0^{l,b} = \delta(l)$ .

We now show that  $\ell = a$ . To see this, assume  $\ell > a$ . Then, since  $\tilde{w}_{p_n}^{a,x_n} \succ \delta(l)$ , there exists a binary lottery  $\tilde{w}_{p_n^*}^{a,x_n^*}$  such that  $l < x_n^* < x_n$ , and  $\tilde{w}_{p_n^*}^{a,x_n^*} \sim \delta(l)$ . Let  $x_{n+1}^*$  and  $p_{n+1}^*$  be defined by  $\tilde{w}_{1-p_{n+1}^*}^{x_{n+1}^*,b} \approx \tilde{w}_{p_n^*}^{a,x_n^*}$ . Since  $\{\tilde{w}_{1-p_n}^{x_n,b}\}_{n \in \mathbb{N}^*}$  converges towards  $\delta(l)$ , there exists an integer  $N$ , such that  $m \geq N \Rightarrow l \leq x_m < x_{n+1}^*$  and  $p_m \geq p_{n+1}^*$ . This implies that  $\tilde{w}_{1-p_{n+1}^*}^{x_{n+1}^*,b}$  should be preferred to the  $\tilde{w}_{1-p_m}^{x_m,b}$ s and, consequently, that  $\delta(l)$  should be preferred to the  $\tilde{w}_{1-p_m}^{x_m,b}$ s, that contradicts the fact that  $\{\tilde{w}_{1-p_n}^{x_n,b}\}_{n \in \mathbb{N}}$  is decreasing and converges towards  $\delta(l)$ . Hence  $\ell = a$  and  $\{S_n\}_{n \in \mathbb{N}}$  converges towards  $\Sigma = \pi u(w)$ . As a consequence, equality (14) is checked.  $\square$