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# Learning by Trading in Infinite Horizon Strategic Market Games with Default*

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**Abstract**

We study the consequences of dropping the perfect competition assumption in an infinite horizon model with infinitely-lived traders and collateralized assets together with asymmetric information among players and incomplete monitoring. Trading assets is not only a way to hedge oneself against uncertainty and to smooth consumption across time: It also enables learning information. We focus on learning equilibria, at the end of which no player has incorrect beliefs because they have taken time to update their prior belief. We prove a partial Folk theorem à la Wiseman (2012) of the following form: For any function that maps each state of the world to a sequence of feasible and strongly individually rational allocations, and for any degree of precision, there is a perfect Bayesian equilibrium in which patient players learn the realized state with this degree of precision and achieve a payoff close to the one specified for each state. When the number of traders grows to infinity, the resulting equilibrium set sharply contrasts with that of standard perfectly competitive equilibria.

**Keywords and Phrases:** Strategic Market Games, Infinite Horizon, Incomplete Markets, Collateral, Incomplete Information, Learning, Adverse Selection

**JEL Classification Numbers:** C72, D43, D52, G12, G14, G18

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# 1 Introduction

A central question for economic theory is the kind of distortions imperfect competition introduces in markets. Arguably, as soon as market participants have market power, prices not only reflect the scarcity of assets relative to the participants' beliefs and preferences but also convey information about their strategic behavior. Knowing this, players can therefore manipulate prices in order to communicate among each other, and of course can lie, bluff or transmit only part of the truth they know. On the other hand, it is well known that market incompleteness also introduces some kind of inefficiency. The most important reason for missing markets, in practice, is private information. Individuals know more about their characteristics, risks and strategies than the market can observe. Despite an important literature on this topic, efficiency properties of economies with private information are not well understood, even less when the lack of information interacts with the strategic power to manipulate prices. In this paper, we investigate the efficiency properties of a large class of economies with private information, those with adverse selection and moral hazard, where individuals have hidden characteristics (known only to their owner) and take privately observed actions affecting their endowments.

For this purpose, we study a strategic market game with an infinite horizon and collateralized assets populated by infinitely-long lived players sharing heterogeneous beliefs, where only prices are publicly observed along the play. Traders start the game with the same prior, but along the play they gather different information (including private information about their characteristics) so that in general, information will be asymmetric among them. Information transmission and strategic manipulation of endogenously formed prices then become the key issue in the analysis of the game. Belief heterogeneity has already been studied in the pioneering works by Sandroni (2000) and Blume and Easley (2006). There, however, it is shown that, within complete markets, agents who learn slowly disappear so that, in the long-run, markets are only populated with agents sharing homogeneous beliefs. Indeed, asymmetries in information induce players to take large bets under frictionless complete financial markets, which enable them to leverage their future wealth. Consequently, after having been driven out of the markets, they have no influence on asset prices and trades in the long run. In order to study the long-run effects on prices and trades of strategic learning, we therefore consider markets with collateral constraints.[1] Indeed, as already observed by Cao (2011), this friction prevents agents with heterogeneous beliefs from being driven out of the market, whatever being the speed at which they learn.

In strategic market games[2], players send bids and offers to the market and a central clearing house answers by computing market clearing prices. Since they are finite in number, players can manipulate their opponents' information by influencing publicly announced prices. Along every equilibrium path considered in this paper, players with incorrect beliefs will possibly learn the state of the world (hence better forecast their future payoffs) through coordinated experimentation by trying different

[1] Collateral requirements for financial assets are introduced as in Geanakoplos and Zame (2007) and the related literature.

[2] See Giraud (2003) for an introduction.

action profiles, observing the resulting prices and updating their beliefs about the region of the event tree where they are currently located. Financial assets therefore turn out to play a double role: On the one hand they serve as means for reallocating one's resources in face of risky future events, on the other hand they can have a function analogous to that of "arms" in the multiarmed bandit problem (Rothschild, 1974) in the following sense. Each player's prior beliefs about the return distribution of a portfolio induce subjective payoff expectations for each asset, but the asset with the highest subjective expected payoff may not be the best one to purchase: A trader may prefer to sacrifice expected returns in the short run to gain some information that will prove useful in the long run. Since there are several traders meeting on the same market, however, the situation may become more intricate, in general: To be fruitful, experimentation has to be somehow coordinated. This coordination requires each cooperative player to be rewarded. Therefore, making information public will have a social cost, in general.

In addition to this first issue, it turns out that players can infer information from market transactions in a more subtle way than by just observing (potentially manipulated) prices. When entering markets in a given node of the event tree, indeed, players do not know the current state of nature in general. Therefore, although each of them knows the contract governing each financial asset, and despite the fact that current prices are publicly announced, the actual payoff of any given asset $j$ remains unknown to any trader $i$, unless she effectively trades this very asset. By observing the actual final payoff of the assets she trades, player $i$ will be able to revise her beliefs. This learning process, however, strikes a major rock: If asset $j$ defaults,[3] none of these two contractants will ever be able to observe the actual payoff of asset $j$. All they know is that this payoff should have been at least as large as what was effectively paid. As a consequence, a trader may prefer to sacrifice expected returns in the short run by enduring default in order to *prevent her opponents* from gaining some information that will prove useful to them in the long run. And in the same way as learning by observing prices is complexified by the very fact that several traders are simultaneously manipulating prices, strategic default is made more complicated by the fact that prices (whose levels will imply whether default occurs or not) are co-determined simultaneously by all the traders.

The piece of good news provided in this paper is that as long as it is compatible with our key Informativeness Assumption (IA, to be described in section 5 below), the possibility of default does not prevent investors from asymptotically learning the state of the world. To put it differently, the Shapley-Shubik trading mechanism provides a sufficiently rich microstructure[4] so that despite price manipulation, infinite-horizon

[3]In this paper, default is not strategic in the following sense: the seller of an asset automatically defaults as soon as the market value of the collateral falls below the promised return of the asset. This is a standard assumption for perfectly competitive general equilibrium models with collateral as in Geanakoplos and Zame (2007), Kubler and Schmedders (2003), Araujo et al. (2005) or Cao (2011). As a consequence, if one seller of asset $j$ defaults, so do all its sellers. In this sense, we can speak of "the default of an asset". A player, however, can strategically induce a default by selling collateral so as to reduce its spot price.

[4]The most obvious one is that the analog of the "minimax" strategy (which consists here in preventing a deviator from purchasing more than $\varepsilon > 0$ of each good) can be made independent from the identity of the deviator. This key property will free us from the need to rely on Blackwell

incomplete markets equilibria turn out to be fully revealing in the precise sense given by our main result: Every feasible stream of allocations and prices can be approximately implemented by an asymptotically fully revealing perfect Bayesian equilibrium as the discount factor goes to one. Somewhat more precisely, for any function that maps each state of the world to a sequence of feasible and strongly individually rational allocations (precise definitions are given in section 3), and for any degree of precision, there is a perfect Bayesian equilibrium in which patient players learn the realized state with this degree of precision and achieve a payoff close to the one specified for each state. Although the formulation of our theorem is very close to the one first introduced by Wiseman (2012), our result cannot be deduced from this paper. There indeed, perfect monitoring allows players to observe the actions of their opponents, hence to condition their own behavior on this observation. Here, by contrast, we assume that quantities traded cannot be observed: Only prices can convey public information, while, whenever exchanged, financial asset payoffs provide some additional insight to their active traders. This forces us to enter into the machinery of the Shapley-Shubik game in order to prove our conclusion.

Provided every investor is patient enough, our result says that no player with incorrect beliefs stays on the market in the long run. This confirms Friedman's (1953) hypothesis (saying that markets are populated, in the long run, with smart investors sharing correct beliefs) but with a completely different argument: although default is compatible with equilibrium, asymptotically, all the traders become smart along every Bayesian equilibrium path, not because the others were eliminated from the market but because they have taken time to cleverly update their prior belief.

The consequences of this finding are twofold on the front of efficiency: On the one hand, we show that price manipulation and learning enable players to coordinate on equilibria that are *ex post* second-best efficient — in particular, they are more efficient than many (possibly only partially revealing) perfectly competitive equilibria. On the other hand, however, and despite the asymptotic full revelation of private information, players may as well coordinate on strongly inefficient perfect Bayesian equilibria. Our result can thus be viewed as a strategic counterpart to the general equilibrium literature with real assets (but no default), where generically every equilibrium is fully revealing (Radner, 1979, Duffie and Shafer, 1985). In addition to the deep difference between our imperfectly competitive approach and the perfect competition hypothesis, a careful reading of our proof, however, shows that our result goes through in the nominal asset case as well.[5] Therefore, from the strategic point of view adopted in this paper, there is no essential difference between real and nominal assets. This contrasts with the negative results obtained in the perfectly competitive general equilibrium literature with incomplete markets of nominal assets (see Rahi, 1995 and the references therein).

To recover perfect competition from the imperfectly competitive setting studied here, a familiar way consists in increasing the number of replica per player. This is where the paper ends up with a surprise: Applying our main result to the $k$-fold replication of our strategic market game, we get that at the limit (as $k \to$

approachability (Blackwell, 1956).

[5]The proof is actually even simpler. This is why we have treated the real asset case.

$\infty$), there are still plenty of perfect Bayesian equilibria exhibiting a large variety of efficiency properties (although each individual becomes asymptotically negligible). This contrasts with the "perfectly competitive case" studied by Cao (2011). There, indeed, it is shown that the presence of heterogeneous beliefs together with collateral requirements lead to additional leverage and asset price volatility (relative to the complete markets economy). By contrast, here, we show that virtually any kind of volatility (in the allocation of commodities and assets) is compatible with the imperfectly competitive equilibrium condition — and this remains so even when the number of replica grows to infinity. Therefore, the limit benchmark obtained by letting the weight of each price maker shrink to zero leads to conclusions that differ sharply from a direct price-taking assumption. This is due to the fact that whenever perfect competition is understood as an asymptotic notion, the monitoring (of private signals and public prices) sharply changes the picture of equilibria: By conditioning their behavior on their observation, players' may diverge deeply from the perfectly competitive behavior. To put it differently, the asymptotic approach of perfect competition reveals that this very notion not only rests on the price-taking assumption but also on some no-monitoring restriction.[6]

The paper is organized as follows: First, in Section 2 we describe the Markov strategic market game with collateral. Then, Section 3 focuses on a particularly important subclass of allocations that plays a key role in the sequel. Afterwards in Section 4 we analyze an example where most of the strategic issues at stake can be easily illustrated. The next section proves a (partial) Folk theorem with incomplete information. Section 6 draws from the previous theorem a non-convergence result for the $k$-fold replicated game. The last section presents the conclusion.

# 2 The Markov Strategic Market Game with Collateral

The underlying environment is a standard economy with infinitely-lived investors (see Kubler and Schmedders, 2003, or Cao, 2011).

## 2.1 The Markov Structure

### The environment

Time is discrete. In each period, $t$, the state of nature is chosen using a Markov transition matrix, $\omega$, with a finite set of possible states of nature $\mathcal{S} = \{1, ..., S\}$. A node $z \in \mathbf{D}$ can be interpreted as a date-event pair $(s_{t-1}, s)$, where $t \geq 1$ is the minimal length of a walk between the root of the event tree, denoted by $z_0$, and $z$, $s_{t-1} \in \prod_{t'=1}^{t-1} \mathcal{S}$ is the sequence of realizations of the state of nature up to $t-1$ and $s \in \mathcal{S}$ is the last state in $t$. Let $\tau(z)$ be the time at which node $z$ is reached, i.e.

[6] This lack of upper-hemi-continuity of the equilibrium correspondence, when dealing with strategic market games, was already observed in Giraud and Stahn (2003) and Giraud and Weyers (2004) in different contexts.

$\tau : \mathbf{D} \to \mathbb{N}$ such that $z = (s_{t-1}, s) \mapsto t$. The unique predecessor of $z$ is denoted by $z^- = (s_{t-2}, s')$.[7] The set of immediate successors of $z$ is $z^+$. For any node $z \in \mathbf{D}$, the set of all nodes that are preceded (resp. strictly preceded) by $z$ is denoted by $\mathbf{D}(z)$ ($\mathbf{D}(z)^+$). Given some Markov chain $\omega$, $\mu_\omega \in \Delta(\mathcal{S})$ is an *invariant measure* of $\omega$ if $\mu_\omega(s) = \sum_{s'} \omega_{s's} \mu_\omega(s')$ for all $s \in \mathcal{S}$. We assume that the Markov chain $\omega$ is **irreducible** and **aperiodic**.[8] We furthermore assume throughout the paper that every state $s \in \mathcal{S}$ is positive recurrent.[9] Therefore, $\omega$ admits a unique invariant measure.

A state of the *world* corresponds to the transition matrix, $\omega$, of dimension $S \times S$ that is chosen once and for all at time 0 before the start of the play. We assume that there are **finitely** many states of the world, $\omega \in \Omega$.

### Consumption goods and financial assets

Let us consider a pure exchange economy $\mathcal{E}$ with a finite set, $\mathcal{N} = \{1, ..., N\}$, of investors, $L$ consumption goods, indexed by $\ell$, and $J$ short-term real assets, indexed by $j$. A financial asset $j \in \mathcal{J} := \{1, ..., J\}$ is characterized by a tuple $(z^j, A_j, C_j)$ consisting of three elements: an issuing node, promised deliveries and collateral requirements. The issuing node (a node in the tree $\mathbf{D}$) is denoted by $z^j$. We only consider short-term assets: The promised amount of goods is described by a function $A_j : \mathbf{D} \to \mathbb{R}^L_+$ such that $A_j(z) = 0$ for all $z \in \mathbf{D} \setminus (z^j)^+$ (i.e., for other nodes before the issuing node and two or more periods after the asset was issued, the promise is zero). For $z' \in (z^j)^+$, the promise $A_j(z')$ is the bundle of goods that a seller of asset $j$ promises to deliver to a buyer of asset $j$ in the next period following the issuing node $z^j$. The delivery, $p_z \cdot A_j(z)$, is assumed to be made in fiat money using spot prices, $p_z \in \mathbb{R}^L_+$. The vector $C_j \in \mathbb{R}^L_+$ is the amount of collateral needed at the issuing node, $z^j$, in order to back up the promised delivery $A_j$. Only consumption goods can serve as collateral.[10] Every commodity lives at most two periods: Either it is consumed at the very period it enters into the economy (as initial endowment) or it is stored as collateral and consumed one period later. In the second period of a collateral's life, it may be traded by its owner and consumed by another player.[11] For simplicity, at each node $z \in \mathbf{D}$ the same finite number of financial assets is issued.

[7] We define $s_{-1} = \emptyset$.

[8] A state $s \in \mathcal{S}$ has period $k$ if any return to state $s$ must occur in multiples of $k$ steps. If $k = 1$, state $s$ is said to be aperiodic. If every state $s \in \mathcal{S}$ is aperiodic, $\omega$ is said to be *aperiodic*. The Markov chain $\omega$ is *irreducible* if it is possible to connect every state $s \in \mathcal{S}$ with any other state $s' \in \mathcal{S}$ with positive probability.

[9] A state $s$ is recurrent if, given that the chain starts in $s$, it will return to $s$ in finite time with probability one. $s$ is *positive recurrent* if, in addition, the expectation of this hitting time is finite.

[10] i.e., we do not introduce securities that are backed by other securities: Pyramiding is not allowed.

[11] We could allow for a longer life expectancy of a collateral, say of length $K$, but at the cost of cumbersome notations. We thus take $K = 1$.

## 2.2 The Strategic Market Game with Collateral

### The one-shot economy

Every player $i \in \mathcal{N}$ is characterized by a twice continuously differentiable, strictly increasing and concave per node utility function $u_z^i : \mathbb{R}_+^L \to \mathbb{R}$ and a non-zero initial endowment in consumption goods $w_z^i \in \mathbb{R}_+^L \setminus \{0\}$ at every node $z \in \mathbf{D}$. We assume that $(u_z^i(\cdot))_z$ are uniformly bounded below for all individuals $i$. Therefore, without loss of generality suppose $u_z^i(0) = 0$. Moreover, we assume that individual endowments are uniformly bounded above by some $\overline{w}$, across individuals and periods. Initial holdings of assets are 0. We denote by $\mathcal{E}_z = \langle w_z^i, u_z^i(\cdot), (z^j, A_j, C_j)_{j \mid z^j = z} \rangle$ the finite-dimensional *one-shot economy* at node $z$.

The remaining part of this subsection is devoted to the description of the strategic market game with collateral played in the one-shot economy $\mathcal{E}_z$ at node $z$. Afterwards the infinite horizon strategic market game is introduced and the overall utility function is defined.

### The trading mechanism

The trading microstructure is given by a market game à la Shapley-Shubik (see Shapley and Shubik, 1977). Each individual places for every consumption good $\ell \in \mathcal{L}$ at every node $z \in \mathbf{D}$ a bid $b_{z,\ell}^i$ and an offer $q_{z,\ell}^i$. The bid $b_{z,\ell}^i$ signals how much (in terms of fiat money) player $i$ is willing to pay for the purchase of good $\ell$, and the offer $q_{z,\ell}^i$ (in terms of physical commodities) is the amount she wants to sell. The price of good $\ell$ is given by:

$$p_{z,\ell} := \begin{cases} \dfrac{\sum_{i=1}^N b_{z,\ell}^i}{\sum_{i=1}^N q_{z,\ell}^i} & \text{if } \sum_{i=1}^N q_{z,\ell}^i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

A trading post without trade is said to be closed.[12] Similarly, at every node $z \in \mathbf{D}$ each player places a bid $\beta_{z,j}^i$ stipulating the amount of money she is ready to spend in buying asset $j$ and offers for sale $\gamma_{z,j}^i$ units of this very asset. The asset's price is:

$$\pi_{z,j} := \begin{cases} \dfrac{\sum_{i=1}^N \beta_{z,j}^i}{\sum_{i=1}^N \gamma_{z,j}^i} & \text{if } \sum_{i=1}^N \gamma_j^i(z) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

When the promises are settled, a seller of the financial asset $j \in \mathcal{J}$ compares the value of the promise with the value of the collateral and pays back the minimal value:

$$D_{z',j} := \min \{p_{z'} \cdot A_j(z'), p_{z'} \cdot C_j\} \tag{D}$$

[12] By defining the price as zero when there is no offer on the market we follow Amir et al. (1990, p.128), Postlewaite and Schmeidler (1978, p.128), Peck et al. (1992, p.275) or Giraud and Weyers (2004, p.474), for example.

at node $z' \in (z^j)^+$. Hence, whether player $i$ defaults or not is not the outcome of a deliberate decision of $i$ alone, but depends upon the commodity price $p_{z'}$, which is strategically determined by all players' bids and offers posted at node $z' \in (z^j)^+$.[13]

### Feasible bids and offers

At every node $z \in \mathbf{D}$ and for every financial asset, player $i$ needs to own the required amount of collateral, which depends on the quantity of the asset offered for sales and *not* on the net trades.[14] Assuming player $i$ offers to sell $\gamma^i_{z,j}$ units of asset $j$ at node $z$, then she needs to store $\gamma^i_{z,j} C_j \in \mathbb{R}^L_+$ as collateral.[15] Feasible bids and offers must satisfy the following two constraints for all commodities $\ell$:

$$\sum_{j=1}^{J} \gamma^i_{z,j} C_{j\ell} \leq w^i_{z,\ell}, \tag{$F1z$}$$

$$q^i_{z,\ell} \leq \sum_{j=1}^{J} \gamma^i_{z^-,j} C_{j\ell} + \Delta(F1z) \tag{$F2z$}$$

where $\Delta(F1z)$ stands for the difference between the right-hand side and the left-hand side of $(F1z)$. Inequality $(F1z)$ says that the collateral stored by $i$ at node $z$ must be taken out of her initial endowments. In particular, the collateral cannot consist of commodities that are already inherited from the past as collaterals. Condition $(F2z)$ says that the offered amount of goods plus the amount of goods that must be stored as a collateral cannot exceed the initial endowment of player $i$ at node $z \in \mathbf{D}$ plus the collateral that was put aside in the previous period. Of course, we impose:

$$q^i_{z,\ell}, b^i_{z,\ell}, \beta^i_{z,j}, \gamma^i_{z,j} \geq 0 \quad \text{for all } \ell \in \mathcal{L},\ j \in \mathcal{J} \tag{$F3z$}$$

### The budget constraint

Player $i$ also faces the following budget constraint on fiat money when placing bids and offers:

$$\sum_{\ell=1}^{L} b^i_{\zeta,\ell} + \sum_{j=1}^{J} \beta^i_{\zeta,j} \leq \sum_{\ell=1}^{L} p_{\zeta,\ell} q^i_{\zeta,\ell} + \sum_{j=1}^{J} \pi_{\zeta,j} \gamma^i_{\zeta,j} + \sum_{j=1}^{J} \left(\theta^i_{\zeta^-,j} - \varphi^i_{\zeta^-,j}\right) D_{\zeta,j} \tag{$*^i_z 1$}$$

for all $\zeta \leq z$ where $\theta^i_{\zeta^-,j}$ denotes the final asset purchases and $\varphi^i_{\zeta^-,j}$ the asset sales at node $\zeta^-$ (as will be defined below). Thus, by condition $(*^i_z 1)$ the total value of bids cannot exceed the amount of money player $i$ can get given her sales and given

[13] The collateral is modeled following for instance Geanakoplos and Zame (2007), Kubler and Schmedders (2003), Araujo et al. (2005) or Cao (2011).

[14] As discussed in Dubey and Geanakoplos (2003), netting before imposing the collateral requirement would suppress any constraint on the size of short sales. While this impairs the very existence of a competitive equilibrium (cf. Hart, 1975), it would make the proof of our partial Folk theorem only easier.

[15] Later, on page 9 when defining the final allocation in consumption goods, the collateral requirement is taken using the final asset sales, denoted by $\varphi^i_{z,j}$ and not directly on the offers $\gamma^i_{z,j}$.

the dividends received from her portfolio, $\theta^i_{\zeta^-,j} - \varphi^i_{\zeta^-,j}$. If $(*^i_z 1)$ is violated, say at node $z$, individual $i$ is removed from the game for all subsequent nodes $\mathbf{D}^+(z)$, and all her goods are confiscated forever.[16] The following condition will also be helpful: For every $i$:

$$\text{Either } \sum_{k\neq i} \gamma^k_{z,j} \neq 0 \quad \text{or} \quad \sum_{k\neq i} \beta^k_{z,j} \neq 0, \tag{$*^i_z 2$}$$

which says that there is at least one other individual on the bidding or on the offering side of the financial markets to trade with $i$.

### Final allocations

After trading took place, player $i$'s holdings of asset $j \in \mathcal{J}$ are given by her sales

$$\varphi^i_{z,j} = \begin{cases} \gamma^i_{z,j} & \text{if } (*^i_z 1) \text{ and } (*^i_z 2) \text{ hold} \\ 0 & \text{otherwise} \end{cases}$$

and her purchases

$$\theta^i_{z,j} = \begin{cases} \frac{\beta^i_{z,j}}{\pi_{z,j}} & \text{if } (*^i_z 1) \text{ and } (*^i_z 2) \text{ hold, } \pi_{z,j} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, player $i$'s allocation of good $\ell \in \mathcal{L}$ available for consumption at the end of the current period at node $z$, is

$$x^i_{z,\ell} = \begin{cases} w^i_{z,\ell} + \sum_{j=1}^J \varphi^i_{z^-,j} C_{j\ell} - q^i_{z,\ell} + \frac{b^i_{z,\ell}}{p_{z,\ell}} - \sum_{j=1}^J \varphi^i_{z,j} C_{j\ell} & \text{if } (*^i_z 1) \text{ holds, } p_{z,\ell} > 0 \\ w^i_{z,\ell} + \sum_{j=1}^J \varphi^i_{z^-,j} C_{j\ell} - q^i_{z,\ell} - \sum_{j=1}^J \varphi^i_{z,j} C_{j\ell} & \text{if } (*^i_z 1) \text{ holds, } p_{z,\ell} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 2.1** Condition $(*^i_z 2)$ emerges naturally once collateral requirements are introduced. Suppose, indeed, that individual $i$ is the only one who wants to trade on the financial markets, i.e., $\sum_{k\neq i} \gamma^k_{z,j} = \sum_{k\neq i} \beta^k_{z,j} = 0$. Absent condition $(*^i_z 2)$, this individual could open the markets by bidding and offering strictly positive amounts of assets. By doing so, every player could store some collateral until the next period just by trading "with herself" today. We therefore require the collateral condition to be in force only when there is effective trade between player $i$ and any other trader.

[16] Therefore, for simplicity, we do not consider refinements of the solvency constraints such as the not-too-tight constraints aiming, among other things, at capturing reputation effects (see, e.g., Martins-Da-Rocha and Vailakis, 2011).

## 2.3 The Infinite Horizon Strategic Market Game with Collateral

### The timing

The timing of events in the infinite horizon strategic market game is as follows:

(0) The state of the world $\omega \in \Omega$ is chosen (once and forever).

(1) Individuals observe their initial endowments and their current node utility function *privately*.

(2) The strategic market game is played: Bids and offers on commodity and financial markets are submitted.

(3) Resulting prices are *publicly* announced and final allocations are *privately* observed.

(4) Players update their beliefs about the state of the world according to the information revealed from (1) to (3).

(5) The next state of nature is chosen according to $\omega$ and the succeeding period starts at (1).

### Allowable strategies

The action set of player $i$ at node $z$ consists in feasible bids and offers:

$$\mathcal{A}_z^i = \left\{ \left(q_{z,\ell}^i, b_{z,\ell}^i\right)_{\ell\in\mathcal{L}}, \left(\gamma_{z,j}^i, \beta_{z,j}^i\right)_{j\in\mathcal{J}} \in \mathbb{R}_+^{2L} \times \mathbb{R}_+^{2J} \middle| (F1z),\ (F2z),\ (F3z) \text{ hold} \right\}.$$

Notice that $\mathcal{A}_z^i$ depends upon $z$ but not upon $\omega$. Let $\mathcal{A}_z := \prod_{i=1}^N \mathcal{A}_z^i$.[17] The stage payoff of player $i$ at node $z = (s_{t-1}, s)$ is given by the utility $u_z^i(x_z^i)$ she obtains from consumption.

Most importantly, and to keep the anonymity of markets, actions are *not* observed along the play of the game, which contrasts with the setting considered by Wiseman (2012), for example. Prices are publicly observed by every player. At each node, every player also observes her own initial endowment, her utility function, her final allocation and also the returns of the assets present in her portfolio. Together with the public history of prices, these observations constitute the *private history* of player $i$. A *strategy* of player $i$ consists in choosing an action at every node $z \in \mathbf{D}$ as a function of her own private history. Let $H_z^i$ denote the set of possible private histories for individual $i$ at node $z$, given by

$$H_z^i := \left\{ \left( w_{z'}^i, u_{z'}^i(\cdot), p_{z'}, \pi_{z'}, \varphi_{z'}^i, \theta_{z'}^i, \varphi_{z'}^i D_{z'}, \theta_{z'}^i D_{z'}, x_{z'}^i, w_z^i, u_z^i(\cdot) \right) \middle| \forall z' < z \right\}.$$

[17] Note that the definition of an action set includes actions that possibly violate the budget constraint $(*_z^i 1)$ or $(*_z^i 2)$. An alternative would consist in incorporating these constraints into the very definition of a player's strategy set, which would lead to a generalized game as introduced by Debreu (1952) (see also Harker, 1991, or Facchinei and Kanzow, 2010).

The history at the root $z_0$ is given by $H^i_{z_0} = \left\{w^i_{z_0}, u^i_{z_0}(\cdot)\right\}$. Formally, a strategy of player $i$ is a map $\sigma^i : \bigcup_{z\in\mathbf{D}} H^i_z \to \left(\mathbb{R}^L_+\right)^2 \times \left(\mathbb{R}^J_+\right)^2$ such that $\sigma^i(h) \in A^i_z$ for all $z \in \mathbf{D}$ and for all $h \in H^i_z$.

**Remark 2.2** As is well-known, strategic market games exhibit no-trade as a one-shot Nash equilibrium.[18] As we want to prove the analog of a Folk theorem, we shall therefore need some threats that enforce the equilibrium path. Allowing for punishment phases that consist in playing the autarkic Nash one-shot equilibrium *ad libitum* would make the task rather easy. In order to prove that our result does *not* depend upon this kind of trick (hence is robust to whatever refinement would allow to get rid of the autarkic one-shot equilibrium[19]), we shall focus on out-of-equilibrium strategies where players effectively trade. A second reason for not relying on the heavy hammer of autarkic Nash equilibria is that, as already said, in adverse selection problems, the market collapse has been sometimes predicted as being the unique rational consequence of differential information. We refrain from using autarky as a punishment in order to emphasize that our proof does not depend upon such a global market collapse, even as an out-of-equilibrium threat, and even though default is explicitly allowed along the equilibrium path.

### Private *interim* beliefs

At each node $z$, payoffs are determined as follows: Action profile $a_z \in \mathcal{A}_z$ is played; it induces, say, $x^i_z$ as a final allocation for player $i$ — which is observed by $i$ only. Notice that when entering at node $z$, player $i$ may not know for sure that the current node is $z$. Thus, when she takes her action, she considers the expectation of her next payoffs according to her current private belief.

We allow for arbitrary correlation of payoffs in each state across players' utilities, endowments and assets' returns. So player $i$'s belief about player $j$'s private payoff and other higher-order beliefs are unrestricted. Let $\mathbb{P}^i_z(h^i_z) \in \Delta(\Omega)$ denote player $i$'s *private belief* at node $z$.[20] Together with actions that were played up to node $z$, denoted by $\sigma^i|_z$, such a probability $\mathbb{P}^i_z(h^i_z)$ induces a distribution $\mathbf{P}^i_z(h^i_z, \sigma^i|_z)$ (or $\mathbf{P}^i_z(\sigma^i|_z)$ in short) over the random characteristics of the economy to be selected after $z$, i.e., over $\bigcup_{z'>z} \mathcal{E}_{z'}$. In particular, it provides a distribution over $i$'s future payoffs which, by a slight abuse of notations, is also denoted by $\mathbf{P}^i_z(\sigma^i|_z)$.

### Utilities

At each node, individuals maximize her expected, discounted utility using their private *interim* belief and a common discount factor $\lambda \in (0, 1)$.[21] The objective function

[18] See Weyers (2004) for the elimination of this autarkic equilibrium after two rounds of elimination of dominated strategies within the extension of the Shapley-Shubik game introduced by Mertens (2003).

[19] Such a refinement has been proposed, e.g. by Weyers (2004). As a consequence, Giraud and Weyers (2004) Folk theorem with complete information was already formulated so as not to rely on the autarkic threat.

[20] Hereby, $\Delta(\Omega)$ is the set of all probability distributions over the finite set of states of the world.

[21] Allowing for idiosyncratic discount factors would only require notational changes.

of player $i$ is therefore of the form

$$U^i_{\mathbf{D}(z)}(x^i, \sigma^i|_z, \omega) := (1-\lambda) \sum_{z'=(s_{t-1},s)\geq z} \lambda^{t-1} E_{\mathbf{P}^i_z(\sigma^i|_z)}\left[u^i_{z'}(x^i_{z'})\right]$$

for each node $z$.

**The infinite horizon economy**

We denote the infinite horizon economy starting from a certain node $z$, that is not necessarily the root $z_0$, for short *the economy after* $z$, by $\bigcup_{z'>z} \mathcal{E}_{z'}$ and define $\mathcal{E}^\infty$ as the infinite horizon economy starting in $z_0$.

# 3 Feasible Allocations and *Interim* Individual Rationality

An *allocation* $(\bar{x}^i)_{i\in\mathcal{N}}$ is an assignment of bundles of consumption goods to every individual in the economy $\mathcal{E}^\infty$. An allocation $(\bar{x}^i)_{i\in\mathcal{N}}$ in consumption goods is said to be *feasible* if there exists a portfolio $(\bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ and a price system $(\bar{p}, \bar{\pi})$ such that all individual budget restrictions are satisfied for every player $i$ and every node $z$, market clearing holds for every good $\ell \in \mathcal{L}$, every asset $j \in \mathcal{J}$ and every node $z$, and trades in financial assets for every good $\ell \in \mathcal{L}$, every node and every player $i$ are compatible with the collateral constraints: $\sum_{j=1}^{J} \bar{\varphi}^i_{z,j} C_{j\ell} \leq w^i_{z,\ell}$.

Clearly, for every individual $i$, the sequence of payoffs resulting from the consumption of initial endowments is bounded from below by a constant, say $\underline{u}^i$. Define $\underline{u} := \min_{i\in\mathcal{N}} \underline{u}^i$. Since initial endowments are uniformly bounded, the stage-game payoff, $u^i_z(\cdot)$, induced by a feasible allocation is also uniformly bounded above by some $\overline{u}^i$ across all action profiles, all states and all periods. Define $\overline{u} := \max_{i\in\mathcal{N}} \overline{u}^i$. A feasible allocation $(\bar{x}^i)_{i\in\mathcal{N}}$ is said to be *strongly individually rational (*SIR*)*, if for all $z \in \mathbf{D}$ we have $u^i_z(\bar{x}^i_z) > u^i_z(w^i_z)$.

**Remark 3.1** Given a feasible allocation, notice that the SIR property holds node-wise for every state of the world $\omega \in \Omega$. Admittedly, this is a quite strong assumption. But as we do not restrict the beliefs of the players, this assumption will be useful to ensure that the feasible allocations we consider are strictly preferred to the initial endowments, independently of the players' beliefs. Note that the Folk theorem we show is partial in the sense that we consider equilibria with learning where the player's beliefs converge to the correct beliefs as given by the state of the world. There might exist other equilibria without learning.

The following Lemma says that our last two definitions generically describe a non-vacuous subset of allocations in the infinite horizon economy $\mathcal{E}^\infty$, on which, from now on, we shall focus.

**Lemma 3.1** *If the initial endowments $(w^i_z)_i \gg 0$ are Pareto-inefficient in the L-good spot economy at each node $z \in \mathbf{D}$, then the economy $\mathcal{E}^\infty$ admits a strongly individually rational and feasible (*SIRF*, for short) allocation.*

A strategy profile $\sigma := (\sigma^i)_i$ is called *full* if the following holds

$$\sum_{i=1}^N q^i_{z,\ell} > 0, \quad \sum_{i=1}^N b^i_{z,\ell} > 0, \quad \sum_{i=1}^N \gamma^i_{z,j} > 0, \quad \sum_{i=1}^N \beta^i_{z,j} > 0$$

for all $\ell \in \mathcal{L}$, $j \in \mathcal{J}$, $z \in \mathbf{D}$. The next Lemma will prove useful for our main result. It shows that every SIRF allocation can be enforced by means of some adequate (full) strategy (to be defined below). Such a strategy, however, need not fulfill any equilibrium requirement.

**Lemma 3.2** *Let $(\bar{x}^i)_{i\in\mathcal{N}}$ be a* SIRF *allocation. Let $(\bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ and $(\bar{p}, \bar{\pi})$ be the corresponding portfolio and price system. Then $(\bar{x}^i, \bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ can be implemented through the following strategy profile in the sense that, node-wise, the allocation resulting from this strategy profile is arbitrarily close to the allocation $(\bar{x}^i)_{i\in\mathcal{N}}$. Whatever being the past history, play the strategy profile consisting of the following node-wise defined actions:*
*for all $z \in \mathbf{D}$, $i \in \mathcal{N}$, $\ell \in \mathcal{L}$ and $j \in \mathcal{J}$*

$$q^i_{z,\ell} = w^i_{z,\ell} + \sum_{j=1}^J \bar{\varphi}^i_{z^-,j} C_{j\ell} \quad \textit{and} \quad b^i_{z,\ell} = \bar{p}_{z,\ell}\left(\bar{x}^i_{z,\ell} + \sum_{j=1}^J \bar{\varphi}^i_{z,j} C_{j\ell}\right),$$

$$\gamma^i_{z,j} = \begin{cases} \bar{\varphi}^i_{z,j} & \textit{if } \sum_{i=1}^N \bar{\varphi}^i_{z,j} > 0 \\ \frac{\delta}{N} & \textit{otherwise} \end{cases} \quad \textit{and} \quad \beta^i_{z,j} = \begin{cases} \bar{\pi}_{z,j}\bar{\theta}^i_{z,j} & \textit{if } \sum_{i=1}^N \bar{\theta}^i_{z,j} > 0 \\ \frac{\delta}{N} & \textit{otherwise} \end{cases}$$

*with $\delta > 0$ small. Clearly, the above strategies are full.*

**Remark 3.2** Were we to target an allocation that does not always require effective trade on the asset markets, then we could not target the allocation exactly using full strategies. For the details we refer to the proofs of Lemma 3.1 and Lemma 3.2 in the Appendix .[22]

# 4 A Leading Example

The next example exhibits a situation where one informed player trades with an uninformed one, and it is not in the interest of the informed player that her opponent learns the state of the world. Moreover, the informed player can play so as to prevent the uninformed one from learning the state of the world.

Consider an economy with two commodities and two individuals, $L = N = 2$. The commodities are $x$ and $y$. There are two states of the world $\Omega = \{\omega_1, \omega_2\}$

[22] A different approach is to not use full strategies on the asset markets and target the allocations exactly, as done in Brangewitz 2012, Chapter 4.

corresponding to a a transition matrix, and, at each period, also two states of nature $\mathcal{S} = \{s_\mathfrak{a}, s_\mathfrak{b}\}$. The transition matrix associated with $\omega_1$ (resp. $\omega_2$) is:

$$\omega_1 = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix} \quad \text{and} \quad \omega_2 = \begin{pmatrix} \varepsilon & 1-\varepsilon \\ 1-\varepsilon & \varepsilon \end{pmatrix} \quad \text{with } \varepsilon > 0 \text{ small.}$$

Each individual $i$ has a non-zero initial endowment in consumption goods $w_z^i$ at every node $z$ that is identical for both states of nature $s_\mathfrak{a}$ and $s_\mathfrak{b}$. Suppose that $w_{z,x}^2 > w_{z,y}^1$ for every node $z$ and that the individual characteristics of each trader are independent from the state of nature $s$, except that the upper sets of the stage-utility function of player 2 are modified by the selection of $s$ as shown in the next Edgeworth box. Being informed of her current utility function, player 2 knows the state of nature.

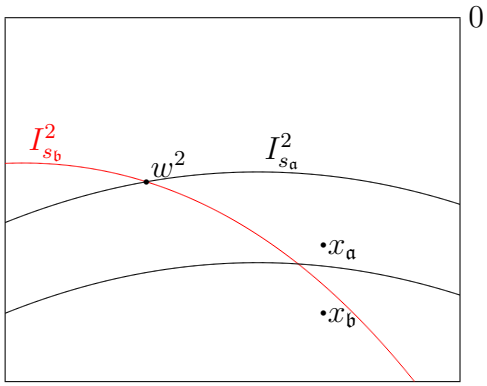



Figure 1: Player 2's random preferences

The hyperbola $I^2_{s_\mathfrak{b}}$ designates the indifference curve of player 2 that crosses 2's initial endowment, $w^2_{s_\mathfrak{b}} = w^2$ in state $s_\mathfrak{b}$. $I^2_{s_\mathfrak{a}}$ is the indifference curve of player 2 in state $s_\mathfrak{a}$ crossing $w^2_{s_\mathfrak{a}} = w^2$.

Suppose, moreover, that a central planner targets a random allocation, depending upon $s$ that ends up at $x_\mathfrak{a}$ whenever $s = s_\mathfrak{a}$, and $x_\mathfrak{b}$ otherwise that is strongly individually rational. The final allocation $x_\mathfrak{a}$ is strictly preferred to $w^2$ by player 2 in state $s_\mathfrak{a}$ but not in state $s_\mathfrak{b}$; $x_\mathfrak{b}$ is strictly preferred by 2 to the initial endowment in both states. Finally, suppose that player 1's prior on $\Omega$ puts a weight $1-\varepsilon$ on $\omega_1$.

A single security can be traded which promises

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the first (resp. second) column stands for the return in state $s_\mathfrak{a}$ (resp. $s_\mathfrak{b}$). The collateral required to sell one unit of this security is $C = (0, 1)$, i.e., one unit of commodity $x$.

By inspecting her own stage utility, player 2 knows the state $s$. Hence, after a sufficiently long time, she can know the state of the world $\omega$ with an arbitrary accuracy by using some statistical tests. From the viewpoint of player 1 on the contrary, the state of the world $\omega$ is a sunspot as it does not affect 1's characteristics. The unique way for player 1 to learn the state of nature (hence, the state of the world)

requires observing the security's return in state $s_{\mathfrak{a}}$. Indeed, in state $s_{\mathfrak{b}}$, whatever being the players' strategies, the asset will default since the value of its pledged collateral always equals the value of their promise. On the contrary, in state $s_{\mathfrak{a}}$ the asset will not default provided the price of commodity $x$ is high enough — thus revealing to traders who own the asset in their portfolio (be it as sellers or as buyers) that they are in state $s_{\mathfrak{a}}$.

Suppose now that the state of nature is $\omega = \omega_2$. The prior belief for both players is such that more weight is put on $\omega_1$. As explained, it is in the interest of player 2 that player 1 continues to think that the state of the world is $\omega_1$. Assume that in $t = 1$ the state of nature $s_{\mathfrak{b}}$ is chosen and in $t = 2$ it is $s_{\mathfrak{a}}$. This sequence is very likely according to $\omega_2$ and pretty unlikely according to $\omega_1$. Player 2 is able to observe that $s_{\mathfrak{a}}$ was chosen. Differently, player 1 cannot observe it by inspecting her private characteristics. If she knew that $s_{\mathfrak{a}}$ was chosen, she would change her belief such that less weight will be put on $\omega_1$ and therefore think that states of nature alternate between $s_{\mathfrak{a}}$ and $s_{\mathfrak{b}}$. This is not in the interest of player 2 as they are supposed to coordinate one period on $x_{\mathfrak{b}}$ and the other one on $x_{\mathfrak{a}}$. Therefore, it is in player 2's interest to keep 1's opinion, i.e., to prevent 1 from updating her belief.

For this purpose, it suffices that player 2 force the default of the security in state $s_{\mathfrak{a}}$ so that player 1 becomes blind after $t = 2$, i.e., $p_{z,y} > p_{z,x}$.[23] To this end, fix the actions $(b^1_{z,\ell}, q^1_{z,\ell})_{\ell=y,x}$ of player 1 at node $z = (s_{\mathfrak{b}}, s_{\mathfrak{a}})$ on the commodity market. Recall that player 1 cannot distinguish between the nodes $z = (s_{\mathfrak{b}}, s_{\mathfrak{a}})$ and $z' = (s_{\mathfrak{b}}, s_{\mathfrak{b}})$ from the observation of his private characteristics and therefore $(b^1_{z,\ell}, q^1_{z,\ell})_{\ell=y,x} = (b^1_{z',\ell}, q^1_{z',\ell})_{\ell=y,x}$. The following action of player 2 induces prices $p_{z,y} > p_{z,x}$ so that the security will default:

$$b^2_{z,x} = q^2_{z,y} = 0,\ q^2_{z,x} > q^1_{z,y} \text{ and } b^2_{z,y} = \frac{b^1_{z,x}}{q^1_{z,x} + q^2_{z,x}} q^2_{z,x}.$$

This action is feasible for player 2 because $w^2_{z,x} > w^1_{z,y}$ and because her budget constraint is binding. The price of commodity $y$ will be given by:

$$p_{z,y} = \frac{b^1_{z,y} + \frac{b^1_{z,x}}{q^1_{z,x}+q^2_{z,x}} q^2_{z,x}}{q^1_{z,y}} = \frac{b^1_{z,y}}{q^1_{z,y}} + \frac{q^2_{z,x}}{q^1_{z,y}} p_{z,x} > p_{z,x}.$$

The strategic challenge raised by this example lies at the heart of the proof of our main result.

# 5 Learning by Trading

As already mentioned, players start with the same prior $\mathbb{P}$ over $\Omega$, but along the play they may (and in general they will) have different *interim* beliefs, depending upon the private information they receive.[24] Even if the results of Wiseman (2012) are not

[23] Prices are formed as usual in strategic market games. The details are explained in the next section.

[24] This is in accordance with the arguments provided by Heifetz (2006) showing that it hardly makes sense within a game-theoretic setting to assume that players start with distinct priors. Of

directly applicable to setting we presented in the previous sections, our main result as well as the general structure of the proof are closely related to Wiseman (2012).

## 5.1 The Main Result

A lesson to be drawn from the leading example of section 4 is that no learning process can be hoped for unless some restrictions are imposed on the economy.

> **Assumption (G).** The set of $L$ consumption goods is partitioned into two subsets, $\mathcal{L} = \mathcal{L}_a \cup \mathcal{L}_c$ with $\mathcal{L}_a \cap \mathcal{L}_c = \emptyset$. Only commodities in $\mathcal{L}_c$ can be used as collateral, and assets' promises deliver only in commodities that belong to $\mathcal{L}_a$.

In words, a commodity cannot serve both as a collateral and as a promise. This assumption is new compared to Brangewitz (2012, Chapter 4) and Giraud and Weyers (2004). That (G) cannot be dispensed with, in general, is shown by the example of the previous section. There, indeed, (G) is violated and the informed player has a mean to prevent her opponent from ever learning the state $\omega$. By contrast, (G) enables to disentangle the prices of the bundle of goods that are promised through financial assets, and the prices of the collaterals. It will turn out that this partition of the set of commodities is sufficient to ensure that during the play of the game, a single player cannot prevent the other investors from learning the true state of the world, $\omega$.

Given the state of the world, $\omega$, a strategy profile, $\sigma = (\sigma^i)$, induces a unique probability distribution, $\mathbf{P}_{\omega,\sigma}$, on the space of sequences

$$(w_z^i, u_z^i(\cdot), p_z(\sigma), \pi_z(\sigma), x_z^i(\sigma), A_j(z))_{i,j,z}. \tag{1}$$

This is the distribution over signals from which players try to infer $\omega$. For any two states $\omega \neq \omega'$, there must be at least some player $i$ and some strategy profile $\sigma = (\sigma^i)_i$ such that the distributions induced by $(\omega, \sigma)$ and $(\omega', \sigma)$ over the sequences (1) differ on a set of positive measure. There is no loss of generality in assuming that a complete sequence of endowments $(w_z^i)_{i,z}$, utility functions $(u_z^i(\cdot))_{i,z}$, prices $p_z(\sigma)$ and $\pi_z(\sigma)$, final allocations $(x_z^i(\sigma))_{i,z}$, and asset returns $(A_j(z))_{j,z}$, jointly identify the state statistically for at least one well-chosen strategy profile, $\sigma$. This does not mean, however that by observing her *own private* sequence of realized individual payoffs, endowments and asset returns, a single trader is able to learn the state of the world whatever the strategy being played. Neither need prices suffice to identify the state by themselves.[25]

The next restriction says the following: Observing the realization of one's (random) initial endowments, (random) stage-utility functions and (strategically determined) final allocations together with *all* the assets' returns, individual final allocations and (public) prices suffices for every trader to learn the true state of the

course, Aumann's (1976) theorem implies that along a play of the game it will not be common knowledge that traders have distinct *interim* beliefs.

[25] When prices are interpreted as public signals, this generality contrasts with Wiseman (2012, p.8, Assumption 1) where the sole observation of public signals suffices to identify the state with no ambiguity.

world in the long run with probability arbitrarily close to one. Needless to say, this assumption is far from being sufficient to guarantee *a priori* that every player will *indeed* learn the true state with arbitrary accuracy: For that purpose, she needs to be able to keep every asset in her portfolio in every period; she may be diverted by the strategic signaling of her opponents; the learning process must remain compatible with the equilibrium conditions, hence should not involve too deep losses. Conversely, abstract from any strategic consideration, this assumption is minimal if learning is to take place along the play of the game.

**Informativeness Assumption (IA)**

(IA-1) For any pair of nodes $(s_{t-1}, s) = z \neq z' = (s_{t-1}, s')$, any player $i$, and any strategy profile $\sigma$ that induces an SIRF allocation at both states ($s$ and $s'$), the vectors of signals, $(w^i_z, u^i_z(\cdot), p_z(\sigma), \pi_z(\sigma), x^i_z(\sigma), A_j(z))_j$ and $(w^i_{z'}, u^i_{z'}(\cdot), p_{z'}(\sigma), \pi_{z'}(\sigma), x^i_{z'}(\sigma), A_j(z'))_j$ differ.

(IA-2) Every $\omega$ is irreducible, aperiodic and admits an invariant measure $\mu_\omega$. Moreover, for any pair $\omega, \omega'$, if $\mu_\omega$ and $\mu_{\omega'}$ are two corresponding invariant measures, then $\omega \neq \omega' \Rightarrow \mu_\omega \neq \mu_{\omega'}$.

(IA-1) requires that for any strategy profile that induces an SIRF allocation, at the end of each period $t$, each player can deduce the node, $z = (s_{t-1}, s)$, at which she was playing, provided she can observe the full signal vector. Of course, this is far from sufficient in order for player $i$ to learn $\omega$, since whether she will be able to observe the full signal vector depends upon the play of the game. (IA-2) is one way of saying that two states of the world induce different distributions over states of nature in the long run. Since we are going to consider patient players, two Markov chains $\omega, \omega'$ that would induce the same asymptotic distribution over signals in the long run should be identified. (IA) cannot be dispensed neither: Consider the situation where the state of the world does not affect the players' private characteristics, and where no asset can be traded. There is no way for players to learn the true state, so no random allocation that depends non-trivially upon the sunspot $\omega$ can be implemented.

**Remark 5.1**

1) (IA) is clearly satisfied when the asset structure is that of a complete set of Arrow securities. In this case, observing assets' returns suffices to identify the Markov chains' realization after each round of trade (even without taking account of prices or of one's private knowledge gained by observing endowments and stage payoffs). After a sufficiently long time, if every trader succeeds in observing every asset's return, the true state of the world will become common knowledge. Notice, however, that even in this polar case, full revelation at a strategic equilibrium has to be proven: Indeed, if one player fails to observe all the assets' returns in certain periods, then she might not draw the right conclusion about which Markov chain is driving uncertainty, so investors cannot coordinate on any state-dependent equilibrium path. On the other hand if, say, only the riskless asset (delivering the same return in every state) is marketed, then observing assets' returns does not provide any information.

2) The differences in information and the signaling aspects of the present work are, for example, related to the job market signaling model of Spence (1973) or the

competitive insurance market considered in Rothschild and Stiglitz (1976) — which, when extended to an infinite horizon, would "naturally" satisfy (IA). However, we depart from the classical principal agent model in as much every individual may act as a seller or a buyer (or both simultaneously), and this on commodity as well as on asset markets. Therefore, we cannot impose, for example, that a seller always is less informed than a buyer or vice versa. Finally, we consider only finitely many players. Our setup therefore sharply differs from the perfectly competitive case studied in the seminal papers by Prescott and Townsend (1984a,b) or, more recently, by Acemoglu and Simsek (2010). In particular, we get a wide range of equilibria including allocation streams that are Pareto-optimal and others that are dominated. Thus, our result stands at distance both from the generic inefficiency obtained by Greenwald and Stiglitz (1984) or Arnott and Stiglitz (1986, 1990, 1991), and from the more positive results obtained by Acemoglu and Simsek (2010).

In our context, a public perfect equilibrium (as defined by Fudenberg et al., 1994) may fail to exist: First a trader's private information influences her expectations about the future behavior of her opponents, hence about future public prices; second, the public history of prices may call for her to play a sequence of actions that she knows from her private signals will yield a payoff below that of her initial endowments (in the nodes that are going to be reached, according to her subjective beliefs). For both reasons, it may be costly for her to ignore her private information, so any best response must, in general, be a private strategy.

A pair $\left((\sigma^i)_{i\in\mathcal{N}}, (\mathbb{P}^i_z(h^i_z))_{i\in\mathcal{N}}\right)$ consisting of a feasible allocation and a system of private beliefs is a *perfect Bayesian equilibrium* (PBE) if

- $(\sigma^i)_{i\in\mathcal{N}}$ is strongly rational given the private beliefs $(\mathbb{P}^i_z(h^i_z))_{i\in\mathcal{N}}$, i.e., starting at any arbitrary node, given the continuation strategies of the other investors, no individual can improve her utility by unilaterally changing her strategy profile given her private beliefs $\mathbb{P}^i_z(h^i_z)$,
- and the private beliefs $(\mathbb{P}^i_z(h^i_z))_{i\in\mathcal{N}}$ are updated via Bayes rule whenever it is possible.[26]

We are now ready to state our main result.

**Theorem 1** *Under (G) and (IA), let $\varepsilon > 0$ and $(x^{*i}[\omega])_{i\in\mathcal{N},\omega\in\Omega}$ be a* SIRF *allocation in consumption goods, and let $\mathbb{P}$ be a prior belief that assigns strictly positive probability to each state of the world. Then there exists a minimal discount factor $\lambda(\mathbb{P}) < 1$ such that for all $\lambda > \lambda(\mathbb{P})$ there is a* PBE *that with probability at least $1-\varepsilon$, conditional on any state $\omega$ being realized, yields a payoff vector within $\varepsilon$ of $\left(U^i_{\mathbf{D}(z_0)}(x^{*i}, \sigma^i|_{z_0}, \omega)\right)_i$. In equilibrium, conditional on $\omega$, each player $i$'s* interim *private belief converges to the truth: $\lim_{t\to\infty} \mathbb{P}^i_{z=(s_{t-1},s)}(h^i_z)[\omega] = 1$ with probability* 1.

[26] Due to our assumptions on the Markov chain, every state of nature is reached with a strictly positive probability. Therefore, given the current state, Bayesian updating is always unambiguous. González-Díaz and Meléndez-Jiménez (2011) discuss the meaning of "whenever it is possible" for general extensive form games with incomplete information. In our special case their notion of a simple perfect Bayesian equilibrium coincides with usual perfect Bayesian equilibrium.

## 5.2 Proof of Theorem 1

The equilibrium path uses "blocks" of $M+T$ periods each, in which traders follow strategies that rely only on the history since the start of the block. They rely on a *truncated belief*, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z) \in \Delta(\Omega)$, defined as follows: Suppose that the block under scrutiny started at node $\bar{z} \leq z$. Given some history, $h^i_z$, consider the truncated history, $h^i_{z\setminus\bar{z}}$, containing only the information delivered from $\bar{z}$ to $z^-$. The truncated belief, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z) \in \Delta(\Omega)$, is the resulting updated belief starting with prior $\mathbb{P}$ at node $\bar{z}$. Therefore, our model can be considered to be Markov in blocks of $M+T$ periods.

Step 1. For each state $\omega \in \Omega$, let $(x^{*i}_z[\omega])_{i,z}$ be a SIRF target allocation with stage payoffs $(v^{*i}_z[\omega])_{i,z} := (u^i_z(x^{*i}_z[\omega]))_{i,z}$. Choose a sequence of payoff vectors $\left((v^{i,n\text{dev}}[\omega])_i\right)_{n\in\mathbb{N}}$ that result from SIRF allocations, and such that: $v^{i,(n+1)\text{dev}}_z[\omega] < v^{i,n\text{dev}}_z[\omega]$ for every integer $n \in \mathbb{N}$, and every $z$ — with $v^{i,0\text{dev}}[\omega] = v^{*i}[\omega]$ for each $i$. These utility levels will be the long-run payoffs after $n$ deviations and can be constructed as:

$$v^{i,n\text{dev}}_z[\omega] := u^i_z\left(x^{i,n\text{dev}}_z[\omega]\right) \text{ with } x^{i,n\text{dev}}_z[\omega] := \rho_n x^{*i}_z[\omega] + (1-\rho_n)w^i_z, \quad \rho_n \in (0,1).^{27}$$

Assume that, for every $n \in \mathbb{N}$ and $z = (s_{t-1}, s) \in \mathbf{D}$:

$$0 < \varepsilon_n < v^{i,n\text{dev}}_z[\omega] - v^{i,(n+1)\text{dev}}_z[\omega] \tag{2}$$

for every player $i$ and every state $\omega \in \Omega$. Notice that $\varepsilon_n$ does not depend upon $z$, while the payoff $v^{i,n\text{dev}}_z[\omega]$ does. The sequences $(\rho_n)_n$ and $(\varepsilon_n)_n$ need to be chosen so as to converge sufficiently rapidly towards $0^+$ (as $n \to +\infty$) for (2) to hold.

Step 2. *The $\delta$-actions.* Every player plays some action on the financial markets so that everybody gets and sells a small quantity, $\delta > 0$, of every security. Consequently, all the commodities that are eligible as collaterals will have to be partially stored. Meanwhile, on the market for consumption goods that serve as a collateral, investors bid very large quantities and offer very small quantities. As a consequence, collateral commodity prices will be large. Let us choose them sufficiently large so that there will be no default along this part of the play. And still, the quantities of commodities that are going to be effectively traded can be made arbitrarily small, as well as the quantities of collaterals they have to put aside because of the trading in securities.

Formally, for node $z = (s_{t-1}, s) \in \mathbf{D}$ in period $\tau(z) = t \in \mathbb{N}$, a $\delta$-action is defined as follows: Let $\delta > 0$ be small. Define the actions on the goods markets by

$$b^i_{z,\ell} := \begin{cases} \bar{b}_\ell > 0 \text{ large}, & \text{for } \ell \in \mathcal{L}_c \\ \frac{\delta}{N} & \text{for } \ell \in \mathcal{L}_a \end{cases} \quad \text{and} \quad q^i_{z,\ell} := \frac{\delta}{N} \quad \text{for } \ell \in \mathcal{L},$$

for all $i \in \mathcal{N}$ and on the asset markets by

$$\beta^i_{z,j} := \frac{\delta}{N} \quad \text{and} \quad \gamma^i_{z,j} := \frac{\delta}{N} \quad \text{for all } j \in \mathcal{J}, i \in \mathcal{N}.$$

It can easily be seen that these actions define feasible bids and offers and that the individual budget constraint is satisfied. The collateral requirement is equal to $\frac{\delta}{N} C_{j\ell}$,

[27] With a slight abuse of notation we omit to denote explicitly the dependence of $v^{i,n\text{dev}}_z[\omega]$ and $x^{i,n\text{dev}}_z[\omega]$ on $\rho_n$. The sequence $(\rho_n)_n$ is assumed to converge to 0.

and hence as $\delta > 0$ is small, condition $(F1z)$ and $(F2z)$ are satisfied. Condition $(F3z)$ is trivially satisfied as well. For the budget feasibility note that if in period $t-1$ everybody already played a $\delta$-action, for the current period at node $z$ the individual budget constraint $(*^i_z 1)$ is easy to verify. Otherwise a transition period is needed as defined in the next step. Moreover condition $(*^i_z 2)$ is satisfied.

Now, what happens if player $i$ deviates from a $\delta$-action profile? She cannot prevent her opponents from observing their own private characteristics. Can she prevent the other players from observing the assets' returns? As she cannot prevent them from trading assets, choosing actions that induce default in all states might stop the learning process of the other players. Acting so as to decrease the price of the collateral commodities while at the same time increasing the price of the non-collateral commodities is the unique way to *cause* default. How can a single player achieve this goal? In order to decrease the price of the collateral commodities at time $t$, she can increase her offers on the commodity market for these goods. By doing so, she is physically constrained by her (finite) initial endowment: This is constraint $(F2z)$. In order to be able to increase the bids for the non-collateral commodities she could first use the money from the additional sales of the collateral commodities and, second, she might have some additional dividends from asset market transactions at time $t-1$. To satisfy the individual budget constraint $(*^i_z 1)$ at time $t-1$, hence to finance the asset purchases in that very period, she needs to make some additional asset sales which are again constraint by the availability of (finite) initial endowments that need to be used to put up for the collateral: this is constraint $(\mathrm{F2}z^-)$ — where $z^-$ is the predecessor of node $z$. Hence, player $i$ can neither increase the bids for non-collateral commodities arbitrarily high nor offer arbitrarily large quantities of collateral commodities. The influence on the price of player $i$ is bounded. Thus, for each node $z$ there exists a lower bound on the bids $\bar{b}_\ell$ in the $\delta$-action profile such that if every trader bids above this bound, player $i$ cannot induce default. From now on, a $\delta$-action will always be understood to be such that every player's bid lies above $\bar{b}_\ell$.

Step 3. *The pre-$M$-transition actions.* If the asset holdings are strictly positive and if players want to switch to a $\delta$-action profile at node $z$, there needs to be transition period to settle the asset market obligations. Otherwise, the $\delta$-action profile might not be budget-feasible i.e., might violate condition $(*^i_z 1)$.
For the pre-$M$-transition period at node $z$, define the following actions:
- on the commodity markets for all $\ell \in \mathcal{L}$, $i \in \mathcal{N}$:

$$b^i_{z,\ell} := \begin{cases} \sum_{j=1}^{J} \theta^i_{z^-,j} C_{j\ell} & \text{for } \ell \in \mathcal{L}_c \\ \frac{N}{\delta} & \text{for } \ell \in \mathcal{L}_a \end{cases} \quad \text{and} \quad q^i_{z,\ell} := \begin{cases} \sum_{j=1}^{J} \varphi^i_{z^-,j} C_{j\ell} & \text{for } \ell \in \mathcal{L}_c \\ \frac{\delta}{N} & \text{for } \ell \in \mathcal{L}_a. \end{cases}$$

- on the asset markets for all $j \in \mathcal{J}$, $i \in \mathcal{N}$:

$$\beta^i_{z,j} := \frac{\delta}{N} \quad \text{and} \quad \gamma^i_{z,j} := \frac{\delta}{N}.$$

The resulting prices are as follows:

$$p_{z,\ell} = \begin{cases} 1 & \text{for } \ell \in \mathcal{L}_c \\ \frac{N^2}{\delta^2} & \text{for } \ell \in \mathcal{L}_a \end{cases} \quad \text{and} \quad \pi_{z,j} = 1.$$

for $\ell \in \mathcal{L}$, $j \in \mathcal{J}$. Choose $\delta$ sufficiently small so that the prices of commodities used for the promises of assets are so large that all assets default in the transition period.

It is easy to verify that the pre-$M$-transition actions satisfy the feasibility constraints $(F1z)$, $(F2z)$ and $(F3z)$. Moreover, the individual budget constraint $(*^i_z 1)$ and condition $(*^i_z 2)$ are satisfied.

From now on, unless otherwise stated, every block of $\delta$-actions will always be preceded by the play of such transition actions.

Step 4. *The post-$M$-transition actions.* After the experimentation block, when a state, $\omega$, has been identified, the investors play actions (according to Lemma 3.2) so as to target a given allocation. This target allocation might require some holdings in certain assets which are not budget feasible given the $\delta$-action played in the last experimentation period (e.g., a player may need to have saved much more money than she did according to the $\delta$-action in order to finance her purchases according to the target allocation). Therefore, we add two periods of post-$M$-transition after the experimentation block where players can settle the asset holdings from the $M$-block (first post-$M$-transition period) and build up the necessary asset holdings for the target allocation (second post-$M$ transition period).

Let the identified state be $\omega$ with target allocation $x^{*i}_z[\omega]$, together with actions, $\varphi^{*i}_z[\omega]$ and $\theta^{*i}_z[\omega]$, on the asset markets. The first post-$M$ transition period is identical to a pre-$M$ transition period (cf. *supra*). The second post-$M$-transition period at node $z$ can be intuitively described as follows: People who have money from asset sales bid it on the goods markets, people who need money offer a tiny little bit of their endowment in order to get money. Commodity prices resulting from this action profile will be high, as only a little bit of commodity is offered. They turn out to be sufficiently high for every player to fulfill her budget constraint. The only point might be that some player is forced to sell a tiny little bit of her initial endowment while the collateral requirement associated with her asset sales requires her whole endowment vector to be collateralized. This would contradict (F2$z$). Thus, players are actually asked to sell a little bit less of assets than would be needed, were they to mimic exactly the target trades in assets. As a consequence, each player will save a small quantity of collateral that can be sold on the commodity market in order to fulfill her budget constraint. It turns out that the quantity of money lost by selling less assets can be compensated by the additional sale of commodities.
More precisely,
- on the commodity markets, play for all $\ell \in \mathcal{L}$:

$$b^i_{z,\ell} := \begin{cases} \frac{1}{\mathcal{L}} \sum_{j=1}^{J} \pi^*_{z,j}[\omega] \left(\varphi^{*i}_{z,j}[\omega] - \theta^{*i}_{z,j}[\omega]\right) & \text{if } \sum_{j=1}^{J} \pi^*_{z,j}[\omega] \left(\varphi^{*i}_{z,j}[\omega] - \theta^{*i}_{z,j}[\omega]\right) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$q^i_{z,\ell} := \begin{cases} -\delta \frac{1}{\mathcal{L}} \sum_{j=1}^{J} \pi^*_{z,j}[\omega] \left(\varphi^{*i}_{z,j}[\omega] - \theta^{*i}_{z,j}[\omega]\right) & \text{if } \sum_{j=1}^{J} \pi^*_{z,j}[\omega] \left(\varphi^{*i}_{z,j}[\omega] - \theta^{*i}_{z,j}[\omega]\right) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- on the asset markets for all $j \in \mathcal{J}$:

$$\beta^i_{z,j} := \pi^*_{z,j}[\omega]\big(\theta^{*i}_{z,j}[\omega] - \eta^i_{z,j}\big) \quad \text{and} \quad \gamma^i_{z,j} := \varphi^{*i}_{z,j}[\omega] - \eta^i_{z,j}$$

for all $j, i$, and where $\eta^i_{z,j} := \sum_\ell \frac{q^i_{z,\ell}}{C_{j\ell}}$ (with the usual convention $1/0 := 0$). Since $x^{*i}_z[\omega]$ is feasible, the asset markets clear, $\sum_{i=1}^N \varphi^{*i}_z[\omega] = \sum_{i=1}^N \theta^{*i}_z[\omega]$. The resulting prices are as follows: $p_{z,\ell} = \frac{1}{\delta}$ and $\pi_{z,j} = \pi^*_{z,j}[\omega]$ for $\ell \in \mathcal{L}$, $j \in \mathcal{J}$. Choose $\delta > 0$ sufficiently small.

It is easy to verify that the transition actions satisfy the feasibility constraints $(F1z)$, $(F2z)$ and $(F3z)$. For the asset trades note that $(F1z)$ holds, as $x^{*i}_z[\omega]$ is feasible. Moreover, the individual budget constraint $(*^i_z 1)$ is satisfied and $(*^i_z 2)$ holds if $\sum_{i=1}^N \varphi^{*i}_z[\omega] > 0$ (otherwise the second transition period is not needed).

Step 5. There are 3 types of blocks: an *equilibrium* block, a *punishment* block and a *post-deviation* block. We now describe the within-block strategies.

The **equilibrium block** has length $M + T$. Suppose the true state of the world is $\omega$. During the first $M$ periods, play $\delta$-actions (with a transition period if this is not the first equilibrium block of the whole play). During the first $M$ periods of an equilibrium block, every trader is able to observe *all* assets' returns and, by combining this information with her own private initial endowments and stage payoffs, updates her prior belief, $\mathbb{P}$. According to (IA), by choosing $M$ long enough, the probability that each player puts a weight larger than $1 - \varepsilon$ on the true state of the world, $\omega$, can be made arbitrarily close to 1, whatever being $\varepsilon > 0$. More precisely, suppose that there exists a positive integer $M$ such that, conditional on any of the finitely many states $\omega \in \Omega$, updating the prior $\mathbb{P}$ with the $M$ signals that result from the $\delta$-action profile yields a posterior truncated probability, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z)$[28], for each player $i$ that puts weight strictly greater than $1/2$ on $\{\omega\}$ with probability at least $1 - \varepsilon$. That such an integer $M$ exists will be proven in Step 6 below.

Let $\hat{\omega}^i$ denote the state given the highest probability by player $i$ under her own belief, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z)$ (ties can be broken arbitrarily). Because of the choice of $M$ (see above), the identified state $\hat{\omega}_i$ is identical across players $i$ with probability at least $(1 - \varepsilon)^N$. Indeed, this would be the probability according to which every player, having observed her own history, will put a weight greater than $1/2$ on the true state, $\omega$, if each history was drawn independently. Even if initial endowments and stage payoffs were probabilistically independent, the assets' returns are certainly not independent. This correlation among histories can only increase the probability above according to which players reach a consensus on the true state.

Let us denote by $\hat{\omega}$ the state on which, with probability at least $1-\varepsilon$, players put the highest posterior probability at the end of the $M$-part of the block.[29] For the remaining $T$ periods of the block, players start with post-$M$-transition actions, and then play the profile that results in a stage payoff $u^i_z\left(x^{*i}_z[\hat{\omega}]\right)$ for $\tau(z) = M+2, ..., M+T$ in state $\hat{\omega}$. The actions are constructed using Lemma 3.2 for every node $z$ with $\tau(z) = M + 3, ..., M + T$. Hence, during the first $M$ periods of an equilibrium block, individuals are learning the true state of the world. In the last $T - 2$ periods, where $T$ is large relative to $M$, the target utility allocation is reached. If player $i$ deviates unilaterally, then the equilibrium block ends immediately, and a punishment block begins in the next period. The lengths, $M$ and $T$, will be chosen more precisely in

[28]The current equilibrium block is supposed to start at time $\tau(\bar{z}) = \bar{t} \in \mathbb{N}$.

[29]Ties can be broken by some arbitrary rule.

steps 6 and 7.

After a deviation, a punishment phase is played, made of a certain number, $P_n$, of punishment blocks, each of length $M+T$, and the end of the current block. The number $P_n$ depends on the number, $n$, of deviations observed. The construction of a **punishment block** is as follows. Players play *throughout* a $\delta$-action profile as defined earlier (preceded by a transition period). This enables to learn during the punishment phase while keeping the size of net trades arbitrarily tiny. If any player unilaterally deviates from the punishment phase, then the punishment block dedicated to the first deviation ends immediately, and a new punishment phase (ending the current block and consisting in $P_{n+1}$ blocks) begins in the next period. After the $P_n$ punishment blocks, if no further deviation has been detected, players switch to a post-deviation block.

Play in a **post-deviation block** is divided into two parts. First, there are $M$ periods of learning using the $\delta$-action profiles, followed by post-$M$-transition actions, and finally there are $T-2$ periods where action profiles are played, such that the target allocation after the $n$th deviation is reached. The target allocation in the $T-2$ last periods of a post-deviation block consists in playing a certain sequence of SIRF allocations. Which allocations are targeted depends on the number of deviations already observed. For example, after the first deviation in the $T-2$ last periods, the profile yielding $v_z^{i,1\text{dev}}[\hat{\omega}]$ in state $\hat{\omega}$ is played, for $z$ with $\tau(z) = M+3, ..., M+T$. The first two periods after the $M$ block are post-$M$-transition actions. Compared to an equilibrium block, a post-deviation block also consists of a learning phase of $M$ periods and a target allocation in the last $T-2$ periods. The difference is that the second sub-block does not target the equilibrium allocation but rather SIRF allocations that are strictly worse than the target allocations of the equilibrium block or the previous post-deviation block.

Step 6. Play begins with an equilibrium block which is followed by a pre-$M$-transition period (for the settlement of assets' obligations) and another equilibrium block if no unilateral deviation was observed. A post-deviation-$n$ block with no additional deviation is followed similarly by a pre-$M$-transition period and another post-deviation-$n$ block. A punishment-$n$ block (i.e., a punishment block devoted to the $n$th deviation) with no unilateral deviation is followed by a post-deviation-$n$ block.

On the equilibrium path, each player's *private* belief, $\mathbb{P}^i_z(h^i_z)$, is derived by Bayesian updating the prior, $\mathbb{P}$, using the information of her *private* history, $h^i_z$. At the same time, each player computes her truncated belief, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z)$ as defined earlier. This belief serves for the identification of the most likely state of the world, $\hat{\omega}$, according to which the allocation $x^{*i}_z[\hat{\omega}]$ is targeted during the last $T-2$ periods of the block. By construction, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z)$ is reset to the prior, $\mathbb{P}$, at the beginning of each block.
For a given $\omega$, let the random variable, $T^\omega_s$, be the first return time to state $s \in \mathcal{S}$: $T^\omega_s := \inf\{n \in \mathbb{N} \mid X^\omega_n = s\}$, where $(X^\omega_n)_n$ stands for the stochastic process (with values in $\mathcal{S}$) corresponding to $\omega$. The number $f^{(n),\omega}_s := \Pr(X^\omega_s = n)$ is the probability that the process returns to state $s$ for the first time after $n$ steps. Since every state $s$ is recurrent, it is easy to prove (and well-known) that the expected number of visits to $s$ is infinite, i.e., $\sum_{n\in\mathbb{N}} p^{(n),\omega}_{ss} = \pm\infty$, where $p^{(n),\omega}_{ks} := Pr(X^\omega_n = s \mid X^\omega_0 = k)$ for any $(k,s) \in \mathcal{S}^2$. Since the Markov chain, $\omega$, is irreducible and such that all the states

in $\mathcal{S}$ are positive recurrent, it admits a unique invariant measure, $\mu_\omega \in \Delta(\mathcal{S})$. The chain $\omega$ being aperiodic, the limit of the expected number, $p_n$, of visits of each state $s \in \mathcal{S}$ verifies: $\lim_{n\to+\infty} p_{ks}^{(n),\omega} = \frac{1}{E[T_s^\omega]} = \mu_\omega(s)$.

If $M$, the length of the experimentation block, is large enough, the probability that for every $i$, $\mathbb{P}^i_{z\setminus\bar{z}}(h^i_z)$ puts the maximal probability on the true state, $\{\omega\}$, at the end of the block can be made arbitrarily large: By observing the realization of their random signals, the players can observe the realization of $X_\omega$ (IA-1), hence can compute the empirical mean corresponding to the expected return time $M_s$ of each state $s$. Hence, they can approximate $p_{ss}^{(n),\omega}$ with arbitrary accuracy. According to (IA-2), two different states of the world, $\omega, \omega'$, will induce different invariant measures, $\mu_\omega, \mu_{\omega'}$. Thus, for $M$ large enough, all the players will be able to distinguish between state $\omega$ and $\omega'$ with probability at least $1-\varepsilon$. As a consequence, all the players will learn the true state with probability at least $1-\varepsilon$. Let us denote by $M_\varepsilon$ the smallest such integer (whose existence was announced in Step 5 above). The crucial observation is that $M_\varepsilon$ is independent from the discount factor $\lambda$, since it concerns only the learning process. From now on, we suppose that $M \geq M_\varepsilon$.

Step 7. It remains to choose $T$ large enough so that each player's welfare loss (with respect to the benchmark $v_z^{*i}[\omega]$) can be compensated by a sufficiently long targeting period of length $T$, provided players are sufficiently patient.

By construction of the $\delta$-actions and by definition of the targeting actions during the $T$-phase of an equilibrium block, the difference between $v_z^{*i}[\omega]$ and the actual payoff that accrues to player $i$ at node $z$ can be made lower than $\varepsilon$ (for $\delta$ sufficiently small). Let us denote by $U^i_{\mathbf{D}(z_0)}(\sigma^*, \omega)$ the final overall payoff induced by the equilibrium strategy, and by $U^i_{\mathbf{D}(z_0)}(x^{*i}, \omega)$, the final payoff induced by our equilibrium target allocation.[30]

During the learning phase (of length $M$) and the post-$M$ transition of two periods of each equilibrium block, the maximal stage-utility loss is $\overline{u}$, while during the targeting phase (of length $T-2$), it is $\varepsilon$. Suppose $T-2 = QM$ for some integer $Q$. One has:

$$U^i_{\mathbf{D}(z_0)}(x^{*i}, \omega) - U^i_{\mathbf{D}(z_0)}(\sigma^*, \omega) \leq (1-\lambda)\sum_{j=0}^{+\infty} \lambda^{jQM}\Big[\frac{1-\lambda^{M+2}}{1-\lambda}\overline{u} + \lambda^{M+2}\frac{1-\lambda^{QM}}{1-\lambda}\varepsilon\Big],$$

where the sum of the right-hand side is taken over the sequence (indexed by $j$) of equilibrium blocks (of length $M+T = (Q+1)M$). Thus,

$$U^i_{\mathbf{D}(z_0)}(x^{*i}, \omega) - U^i_{\mathbf{D}(z_0)}(\sigma^*, \omega) \leq \frac{1-\lambda^{M+2}}{1-\lambda^{QM}}\overline{u} + \lambda^{M+2}\varepsilon.$$

For every $\varepsilon > 0$, there exists some $Q_{\varepsilon,\lambda}$ large enough and some $\lambda_\varepsilon$ close enough to 1, so the right-hand side of the last inequality is lower than $\varepsilon$. From now on, we assume that $Q \geq Q_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$.

[30]The slight abuse of notations in the arguments of the overall utility should not create any confusion.

Therefore, along the equilibrium path, players learn the true state with probability at least $1-\varepsilon$ and their final payoff will be within $\varepsilon$ of the benchmark. It follows that a patient player prefers not to deviate even if the truncated belief after a sequence of misleading signals calls for an action profile that she thinks will give her a very low payoff for the duration of the current block: At the start of the next block, the pseudo-belief will revert to the prior, and with high likelihood, experimentation in the next blocks will reveal the true state of the world and enable the other players to provide her with the equilibrium payoff or to effectively punish her in case of deviation.

Actions may reveal a piece of information about a player's private payoffs. For instance, by deviating, player $i$ may induce a final allocation for player $j$ different from the one that is prescribed at equilibrium. This different allocation may in turn provide $j$ with some information in terms stage payoff that was out of scope with the equilibrium allocation. And even during a punishment phase, a deviator might be tempted to keep talking with her opponents through the manipulation of their commodity allocations. Nevertheless, (IA) implies that as long as they still observe every asset's return, all the players will learn the true state with arbitrary precision *whatever* being their stream of stage payoffs.[31] By manipulating allocations (hence stage payoffs), a player cannot prevent her opponents from eventually learning the true state of the world, $\omega$.

Step 8. It remains to choose $P_n$ (the number of punishment blocks after $n$ deviations) large enough so that no player has any incentive to deviate, neither on the equilibrium path nor off this path, whatever her private belief about $\omega$ or her higher order beliefs (about others' beliefs). For this purpose, we need to guarantee that a post-deviation long-run discounted payoff never exceeds the equilibrium long-run discounted payoff. Suppose that the deviation occurs at node $z'=(s_{t'-1},s')$, that it is the $(n+1)$th deviation observed during the play and that there are no further deviations at later nodes. It will at most yield $\overline{u}^i$ to player $i$. Then, the post-deviation payoff can be made $\varepsilon$-close to the following maximum:

$$
\begin{aligned}
(1-\lambda)\lambda^{t'-1}\Big[\overline{u}^i &+ \sum_{t=2}^{M+T}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[u^i_z\left(w^i_z\right)] + \sum_{k=1}^{P_{n+1}}\sum_{t=k(M+T)+1}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[u^i_z\left(w^i_z\right)] \\
&+ \sum_{k\geq P_{n+1}+1}\Big[\sum_{t=k(M+T)+1}^{k(M+T)+M}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[u^i_z\left(w^i_z\right)] + \sum_{t=k(M+T)+M+1}^{k(M+T)+M+2}\lambda^{t-1}\overline{u} \\
&+ \sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[v^{i,(n+1)\text{dev}}_z[\omega]]\Big]\Big]. \qquad (3)
\end{aligned}
$$

Indeed, the long-run discounted payoff after a deviation consists once of a (maybe) very high payoff from deviating, then the payoff from a punishment during the current

[31] In other words, players need to be able to observe the sequence of stage payoffs resulting from *some* SIRF allocation, plus asset returns and initial endowments. For a patient player, the choice of the *particular* sequence of SIRF allocations is irrelevant.

block plus $P_{n+1}$ punishment blocks lasting $M+T$ periods, and finally the payoff from succeeding post-deviation-$(n+1)$ blocks of $M+T$ periods including possibly a very high payoff in the post-$M$-transition period. On the other hand, since no deviator can prevent her opponents from learning the state of the world with arbitrary precision (even during the punishment phase and whatever being the behavior of the deviator), the reward payoff, $E_{\mathbf{P}^i_z(\sigma^i|_z)}[v_z^{i,(n+1)\text{dev}}[\hat{\omega}]]$, computed with the most likely state, $\hat{\omega}$ (according to the players' truncated belief), can also be made arbitrarily close to $E_{\mathbf{P}^i_z(\sigma^i|_z)}[v_z^{i,(n+1)\text{dev}}[\omega]]$.

By contrast, if the $(n+1)$th deviation did <u>not</u> take place, $i$'s long-run discounted payoff would consist in the payoff from post-deviation-$n$ blocks of $M+T$ periods. Therefore, it would be arbitrarily close to:

$$(1-\lambda)\lambda^{t'-1}\sum_{k\geq 0}\Big[\sum_{t=k(M+T)+1}^{k(M+T)+M}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[u_z^i\left(w_z^i\right)] + \sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[v_z^{i,n\text{dev}}[\omega]]\Big]. \tag{4}$$

Note that here we assumed a payoff of 0 in the post-$M$-transition periods. In order to check whether the difference (4) - (3) is positive, all we need is to ensure that:

$$(1-\lambda)\lambda^{t'-1}\Big[\underline{u}-\overline{u}+\sum_{k=1}^{P_{n+1}}\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}\Big[v_z^{i,n\text{dev}}[\omega]-u_z^i\left(w_z^i\right)\Big] + \varepsilon_n\sum_{k\geq P_{n+1}+1}\Big[\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}\Big]-\overline{u}\sum_{k\geq 0}\Big[\sum_{t=k(M+T)+M+1}^{k(M+T)+M+2}\lambda^{t-1}\Big]\Big]>0$$

Note that since $v^{i,n\text{dev}}[\omega]$ results from a SIRF allocation, we have

$$E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[v_z^{i,n\text{dev}}[\omega]-u_z^i\left(w_z^i\right)]>0$$

for every node $z=(s_{t-1},s)$, and every individual $i$. Let us define

$$g_z:=\min_{i\in\mathcal{N}}E_{\mathbf{P}^i_{z'}(\sigma|_{z'})}[v_z^{i,n\text{dev}}[\omega]-u_z^i\left(w_z^i\right)].$$

It is sufficient to require that:

$$(1-\lambda)\lambda^{t'-1}\Big[\underline{u}-\overline{u}+\sum_{k=1}^{P_{n+1}}\Big[\sum_{t=k(M+T)+M+3}^{(k+1)(M+T)}\lambda^{t-1}g_z\Big] + \varepsilon_n\frac{(1-\lambda^{T-2})}{(1-\lambda)(1-\lambda^{M+T})}\lambda^{(P_{n+1}+1)(M+T)+M+2}-\overline{u}\frac{(1-\lambda^2)}{(1-\lambda)(1-\lambda^{M+T})}\lambda^M\Big]>0.$$

It is easy to see that whatever being the distance, $(1-\lambda)\underline{u}-\left((1-\lambda)+\frac{(1-\lambda^2)}{(1-\lambda^{M+T})}\lambda^M\right)\overline{u}$, and that for every $\varepsilon_n > 0$ and every $\lambda \geq \lambda_\varepsilon$, there exists some integer $P_{n+1}^{\lambda,\varepsilon}$ big enough so that this last inequality is satisfied.

Suppose a deviator keeps deviating. While being punished by $\delta$-actions, the most she can grasp is $\delta$ units of each commodity in each period. Immediately a new punishment starts with a punishment phase at least as long as the one before. As the reward in the post-deviation block declines, continuing deviating becomes even less attractive as the payoff in equation (3).

This completes the proof that it is in no player's interest to deviate from the prescribed equilibrium strategy, be it on the equilibrium path (i.e., whenever no deviation already occurred) or out of the equilibrium path (i.e., after a deviation occurred), provided: $M \geq M_\varepsilon,\ \lambda \geq \lambda_\varepsilon,\ T = QM$ with $Q \geq Q_\varepsilon$, and $\forall n,\ P_n \geq P_n^{\lambda,\varepsilon}$. □

# 6 A Nonconvergence Result

In this section we apply the preceding result to study the efficiency properties of the Nash equilibria of the infinite horizon strategic market game where each player becomes negligible. Postlewaite and Schmeidler (1978) show for the truncated game with finite horizon that if there are sufficiently many traders and if aggregate initial endowments are large enough relative to the number of traders, then any full Nash equilibrium is $\eta$-efficient, with $\eta > 0$ arbitrarily small. Contrary to what is obtained in Postlewaite and Schmeidler (1978), for the one-shot game — but in accordance with the result obtained in Giraud and Weyers (2004) for the finite horizon case-complete information case — here we do not obtain a convergence to "almost" efficient equilibria. Quite the contrary, we easily deduce from our Folk theorem that whatever being the number of replica per player, the set of SIRF allocations remains approximately reachable by (type-symmetric) Bayesian perfect equilibria. Obviously, this is due to the fact that however small the weight of a player is, as long as it is non-zero, this player can influence prices so that the monitoring of prices reveals a relevant information across time — hence opens the door to the whole strategic machinery underlying our preceding result.

For $\eta > 0$, a SIRF allocation $(x^i)_{i\in\mathcal{N}}$ is *$\eta$-efficient* if there does not exist any SIRF allocation that Pareto dominates $(x^i)_{i\in\mathcal{N}}$ in any fictitious economy with the same utility functions and in which the aggregate initial endowment vectors are at most $(1-\eta)\sum_i w^i$ — where $(w^i)_{i\in\mathcal{N}}$ are the initial endowments in the original economy. For the sake of completeness, let us recall the following result stated in the one-shot game associated to the one-period Arrow-Debreu economy, $\mathcal{E}_z = (u^i, w^i)_i$ (at node $z \in \mathbf{D}$).

**Approximate Efficiency Theorem.** (Postlewaite and Schmeidler 1978) *For any positive number $\alpha, \beta, \eta$ any allocation resulting from a full Nash equilibrium in an economy with $u_z^i(\cdot)$ and $w_z^i$ is $\eta$-efficient, where the initial endowments $w_z^i \leq (\beta, ..., \beta)$ for all individuals $i \in \mathcal{N}$ and $\sum_{i=1}^N w_z^i > (N\alpha, ...N\alpha)$ and $N > \frac{16L\beta}{\alpha\eta^2}$.*

In order to be able to compare this result with the properties of our infinite-horizon economy, recall that whenever initial endowments are uniformly bounded, the commodity space of the infinite-horizon economy, $\mathcal{E}^\infty$, can be viewed as the space, $\ell^\infty(\mathbf{D} \times L)$, of uniformly bounded stochastic processes defined over the event tree, $\mathbf{D}$, and taking values in $\mathbb{R}^L$. Let us endow this commodity space with the classical Mackey topology, $\tau(\ell^\infty, \ell^1)$.[32] By contrast wth the Approximate Efficiency Theorem just recalled, we get the following corollary of Theorem 1, whose proof is given in the Appendix:

**Corollary 6.1** *Suppose that (G) and (IA) are in force. Take $\eta > 0$ and an integer $N$, arbitrary. Whatever being the number, $N$, of players, if they are sufficiently patient, if $(w^i)_{i\in\mathcal{N}}$ is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient allocations, then there exists a non-empty, $\tau(\ell^\infty, \ell^1)$-open subset of $\ell^\infty(\mathbf{D})$ consisting of allocations that are not $\eta$-efficient and that can nevertheless be approximated by a* PBE *(in the sense of Theorem 1).*

# 7 Concluding Comments

In this paper we investigated the general properties of perfect Bayesian equilibria in imperfectly competitive environments with incomplete information and partial monitoring. We proved that adding collateral constraints within the rules of trading has an ambiguous effect. As already observed by (Cao, 2011, p.2):

> "Collateral constraints limit the extent to which agents can pledge their future wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market.Consequently all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of those heterogeneous beliefs."

When investors can influence prices, indeed, traders with incorrect beliefs can strategically *learn* the state of the world. We thus provided a partial characterization of learning equilibria, at the end of which no player shares incorrect beliefs — not because they were eliminated from the market (although default is possible at equilibrium) but because they have taken time to update their prior belief. The striking point is that our (partial) Folk theorem provides us with a wide range of equilibria, many of them being first-best efficient, many others being dominated. This, at the very least, suggests that considerable care is necessary in invoking the impact of collateral regulation on the inefficiency of financial markets.[33]

In terms of future research, we see the following directions: First, exploring the situation of a reducible Markov chain (or, possibly, a non-Markovian stochastic process) would certainly provide new challenges. Second, it is well-known that several infinite-horizon games with incomplete information have no Bayes-Nash equilibria

[32] That is, the strongest topology such that $\ell^1(\mathbf{D} \times L)$ is the topological dual of $\ell^\infty(\mathbf{D} \times L)$, see, e.g., Magill and Quinzii (1994).

[33] Cf. e.g.,`http://isda.org/c_and_a/collateral.html`

(not to speak about perfect equilibria) once the discounted utility function is replaced by some limit of the arithmetic mean. Thus, there is, in general, a fundamental discontinuity of the equilibrium set as the discount factor goes to one. Does this discontinuity pervade in the specific situation of a strategic market game? Next, the financial friction adopted in this paper consisted in imposing a collateral constraint together with short sales. Many other frictions have been studied in the literature, such as pledgeable income models (Holmstrom and Tirole, 1997), models of private information and costly state verification (Townsend, 1979) or self-enforcing debt constraints (Kehoe and Levine, 1993). Each of these alternatives can be tested to improve the robustness of our result — or to get a smaller range of equilibrium allocations — with respect to the institutional specification of the financial frictions at play. Next, Shapley-Shubik games are but *one* type of market games. Many others can be thought of (see Giraud, 2003), which all require a specific treatment in order to pave the road for a general treatment. Finally, our Folk theorem shows that imperfect competition can circumvent the inefficiency of incomplete markets but at the cost of a huge indeterminacy of equilibria. Which kind of strategic refinement (or, alternatively, which restriction on the players' rationality) may help solve this coordination issue remains an open question for strategic market games.

# A Proofs

## A.1 Proof of Lemma 3.1

To show Lemma 3.1 for our model, we modify the proof of Giraud and Weyers (2004) slightly. **Proof.** Fix a node $z \in \mathbf{D}$. Since the allocation of initial endowments $(w_z^i)_i$ are Pareto-inefficient in the $L$-good spot economy, there exists a feasible allocation $(\bar{x}_z^i)_i$ that Pareto dominates $(w_z^i)_i$ and satisfies for every good $\ell \in \mathcal{L}$

$$\sum_{i=1}^{N} \bar{x}_{z,\ell}^i = \sum_{i=1}^{N} w_{z,\ell}^i.$$

By the strict monotonicity of the preferences, there exists a feasible allocation $(\bar{x}_z'^i)_i$ such that

$$u_z^i(\bar{x}_z'^i) > u_z^i(\bar{x}_z^i) \quad i = 1, ..., N$$

and

$$\sum_{i=1}^{N} \bar{x}_{z,\ell}'^i = \sum_{i=1}^{N} w_{z,\ell}^i.$$

Since the utility functions are strictly increasing, there exists a hyperplane containing $(\bar{x}_z'^i)_i$ and $(w_z^i)_i$ with a strictly positive price vector $p_z$. Thus, the individual budget restriction

$$p_z \cdot \bar{x}_z'^i = p_z \cdot w_z^i$$

is satisfied and furthermore

$$u_z^i(\bar{x}_z'^i) > u_z^i(\bar{x}_z^i) \geq u_z^i(w_z^i)$$

for all $i \in \mathcal{N}$. This argument holds for all $z \in \mathbf{D}$, which proves the claim. □

## A.2 Proof of Lemma 3.2

To show Lemma 3.2 for our model, we modify the proof of Giraud and Weyers (2004). **Proof.** Since $(\bar{x}^i)_{i\in N}$ is a feasible allocation there exist $(\bar{\varphi}^i, \bar{\theta}^i)_{i\in\mathcal{N}}$ such that the asset markets clear at every node $z \in \mathbf{D}$. For all $j \in \mathcal{J}$ we have

$$\sum_{i=1}^{N} \bar{\theta}_{z,j}^i = \sum_{i=1}^{N} \bar{\varphi}_{z,j}^i.$$

Therefore, if $\sum_{i=1}^{N} \bar{\theta}_{z,j}^i = 0$, then $\sum_{i=1}^{N} \bar{\varphi}_{z,j}^i = 0$ and vice versa.

Using the market clearing condition on the goods markets we obtain from the definition of the actions

$$\sum_{i=1}^{N} q_{z,\ell}^i = \sum_{i=1}^{N} \left( w_{z,\ell}^i + \sum_{j=1}^{J} \bar{\varphi}_{z^-,j}^i C_{j\ell} \right) = \sum_{i=1}^{N} \left( \bar{x}_{z,\ell}^i + \sum_{j=1}^{J} \bar{\varphi}_{z,j}^i C_{j\ell} \right),$$

$$\sum_{i=1}^{N} b_{z,\ell}^i = p_{z,\ell} \sum_{i=1}^{N} \left( \bar{x}_{z,\ell}^i + \sum_{j=1}^{J} \bar{\varphi}_{z,j}^i C_{j\ell} \right).$$

Hence,

$$\bar{p}_{z,\ell} = \frac{\sum_{i=1}^{n} b^i_{z,\ell}}{\sum_{i=1}^{n} q^i_{z,\ell}} = p_{z,\ell}.$$

From the definition of the actions using the market clearing condition on the asset markets we obtain for the asset prices

- for $\sum_{i=1}^{N} \bar{\theta}^i_{z,j} = \sum_{i=1}^{N} \bar{\varphi}^i_{z,j} > 0$

$$\pi_{z,j} = \frac{\sum_{i=1}^{N} \beta^i_{z,j}}{\sum_{i=1}^{N} \gamma^i_{z,j}} = \frac{\bar{\pi}_{z,j} \sum_{i=1}^{N} \bar{\theta}^i_{z,j}}{\sum_{i=1}^{N} \bar{\varphi}^i_{z,j}} = \bar{\pi}_{z,j}$$

- for $\sum_{i=1}^{N} \bar{\theta}^i_{z,j} = \sum_{i=1}^{N} \bar{\varphi}^i_{z,j} = 0$

$$\pi_{z,j} = \frac{\sum_{i=1}^{N} \beta^i_{z,j}}{\sum_{i=1}^{N} \gamma^i_{z,j}} = \frac{\bar{\pi}_{z,j} \sum_{i=1}^{N} \frac{\delta}{N}}{\sum_{i=1}^{N} \frac{\delta}{N}} = \bar{\pi}_{z,j}.$$

The final allocation of sales and of purchases for asset $j \in \mathcal{J}$ are given by

$$\varphi^i_{z,j} = \gamma^i_{z,j} \quad \text{and} \quad \theta^i_{z,j} = \frac{\beta^i_{z,j}}{\pi_{z,j}}.$$

The final allocation of good $\ell \in \mathcal{L}$ available for consumption after trading at node $z \in \mathbf{D}$ is given by

$$x^i_{z,\ell} = w^i_{z,\ell} + \sum_{j=1}^{J} \varphi^i_{z^-,j} C_{j\ell} - q^i_{z,\ell} + \frac{b^i_{z,\ell}}{p_{z,\ell}} - \sum_{j=1}^{J} \varphi^i_{z,j} C_{j\ell}.$$

Therefore,

$$\varphi^i_{z,j} = \begin{cases} \bar{\varphi}^i_{z,j} & \text{if } \sum_{i=1}^{N} \bar{\varphi}^i_{z,j} > 0 \\ \frac{\delta}{N} & \text{otherwise} \end{cases}$$

$$\theta^i_{z,j} = \begin{cases} \bar{\theta}^i_{z,j} & \text{if } \sum_{i=1}^{N} \bar{\theta}^i_{z,j} > 0 \\ \frac{\delta}{N} & \text{otherwise} \end{cases}$$

$$x^i_{z,\ell} = \begin{cases} \bar{x}^i_{z,\ell} & \text{if } \sum_{i=1}^{N} \bar{\varphi}^i_{z,j} = \sum_{i=1}^{N} \bar{\theta}^i_{z,j} > 0 \\ \bar{x}^i_{z,\ell} - \sum_{j=1}^{J} \frac{\delta}{N} C_{j\ell} & \text{otherwise.} \end{cases}$$

It remains to check that the budget constraint $(*_z^i 1)$ for the bids and offers is satisfied:

$$\sum_{\ell=1}^{L} b_{z,\ell}^i + \sum_{j=1}^{J} \beta_{z,j}^i \leq \sum_{\ell=1}^{L} p_{z,\ell} q_{z,\ell}^i + \sum_{j=1}^{J} \pi_j \gamma_{z,j}^i + \sum_{j=1}^{J} \left(\theta_{z^-,j}^i - \varphi_{z^-,j}^i\right) D_{z,j}.$$

Inserting the assumed strategies for $b_{z,\ell}^i$, $q_{z,\ell}^i$, $\gamma_{z,j}^i$ and $\beta_{z,j}^i$ we obtain for $(*_z^i 1)$

- for $\sum_{i=1}^{N} \bar{\theta}_{z,j}^i = \sum_{i=1}^{N} \bar{\varphi}_{z,j}^i > 0$

$$\sum_{\ell=1}^{L} \bar{p}_{z,\ell} \left( \bar{x}_{z,\ell}^i + \sum_{j=1}^{J} \bar{\varphi}_{z,j}^i C_{j\ell} \right) + \sum_{j=1}^{J} \bar{\pi}_{z,j} \left(\bar{\theta}_{z,j}^i - \bar{\varphi}_{z,j}^i\right)$$
$$\leq \sum_{\ell=1}^{L} \bar{p}_{z,\ell} \left( w_{z,\ell}^i + \sum_{j=1}^{J} \bar{\varphi}_{z^-,j}^i C_{j\ell} \right) + \sum_{j=1}^{J} \left(\bar{\theta}_{z^-,j}^i - \bar{\varphi}_{z^-,j}^i\right) D_{z,j}$$

- for $\sum_{i=1}^{N} \bar{\theta}_{z,j}^i = \sum_{i=1}^{N} \bar{\varphi}_{z,j}^i = 0$

$$\sum_{\ell=1}^{L} \bar{p}_{z,\ell} \bar{x}_{z,\ell}^i \leq \sum_{\ell=1}^{L} \bar{p}_{z,\ell} \left( w_{z,\ell}^i + \sum_{j=1}^{J} \bar{\varphi}_{z^-,j}^i C_{j\ell} \right) + \sum_{j=1}^{J} \left(\bar{\theta}_{z^-,j}^i - \bar{\varphi}_{z^-,j}^i\right) D_{z,j}$$

which holds since $(\bar{x}^i)_{i\in\mathcal{N}}$ was assumed to be a feasible allocation. As $(w_z^i)_i \gg 0$, this strategy profile is full. This completes the proof. □

## A.3 Proof of Corollary 6.1

We begin with:

**Lemma A.1** *For any number, $N$, of players,*

> *(i) if $(w^i)_{i\in\mathcal{N}}$ is $\eta$-efficient, then every* SIRF *allocation is $\eta$-efficient.*
>
> *(ii) if $(w^i)_{i\in\mathcal{N}}$ is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient allocations, then the set of* SIRF *allocations contains a $\tau(\ell^\infty, \ell^1)$-open subset that is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient allocations.*

**Proof of Lemma A.1.** Let us first consider the economy, $\mathcal{E}^T$, obtained after having truncated $\mathcal{E}$ at $T$.

(i) Suppose $(w_z^i)_{i\in\mathcal{N},\tau(z)\leq T}$ is $\eta$-efficient in $\mathcal{E}^T$. Let $(x_z^i)_{i\in\mathcal{N},\tau(z)\leq T}$ be a SIRF allocation that is not $\eta$-efficient. Then by the definition of $\eta$-efficiency there exists $y_z = (y_z^i)_{i\in\mathcal{N},\tau(z)\leq T}$ such that

$$u_z^i(y_z^i) \geq u^i(x_z^i)$$
$$\sum_{i=1}^{N} y_z^i \leq (1-\eta) \sum_{i=1}^{N} x_z^i = (1-\eta) \sum_{i=1}^{N} w_z^i$$

for at least one $z \in \mathbf{D}$. Since $x^i_z$ is SIR we have $u^i_z(x^i_z) > u^i_z(w^i_z)$. Hence, $u^i_z(y^i_z) > u^i_z(w^i_z)$ and $\sum_{i=1}^N y^i_z \leq (1-\eta)\sum_{i=1}^N w^i_z$ at $z \in \mathbf{D}$. Therefore, leaving the allocations at nodes other than $z$ unchanged shows that this is in contradiction to the assumption that $w^i$ was $\eta$-efficient.

(ii) Suppose $(w^i_z)_{i\in\mathcal{N},\tau(z)\leq T}$ is not $\eta$-efficient. Take $(y^i_z)_{i\in\mathcal{N},\tau(z)\leq T}$ feasible that strictly Pareto-dominates the initial endowments $(w^i_z)_{i\in\mathcal{N},\tau(z)\leq T}$, as in the proof of Lemma 3.1, and for $z \in \mathbf{D}$ with $\tau(z) \leq T$ let $y^i_{k,z} = \frac{1}{k}y^i_z + \frac{k-1}{k}w^i_z$ for all $i \in \mathcal{N}$. Then $(y^i_{k,z})_{i\in\mathcal{N},\tau(z)\leq T}$ is SIRF for all $k$. Because $(w^i_z)_{i\in\mathcal{N},\tau(z)\leq T}$ is not in the closure of the set of $\eta$-efficient allocations, there exists a $k_0$ such that for every $k > k_0$ we have that $(y^i_{k,z})_{i\in\mathcal{N},\tau(z)\leq T}$ is not in the closure of the set of $\eta$-efficient allocations. Because of the strict concavity of preferences we have $u^i_z(y^i_{k,z}) > u^i_z(w^i_z)$ for all $i \in \mathcal{N}$. Thus, for all nodes $z \in \mathbf{D}$ with $\tau(z) \leq T$ we have that $(y^i_{k,z})_{i\in\mathcal{N}}$ is in the interior of the individual rational and feasible allocations of the spot economy. Since preferences are strictly monotonic, for every feasible allocation belonging to some neighborhood of $(y^i_{k,z})_{i\in\mathcal{N}}$ there is a hyperplane through this allocation and the initial endowment $(w^i_z)_{i\in\mathcal{N}}$ with a strictly positive price vector. Thus, there exists a $r_z$ such that all allocations in the ball, $B\left((y^i_{k,z})_{i\in\mathcal{N}}, r_z\right)$, with radius $r_z$ at center $(y^i_{k,z})_{i\in\mathcal{N}}$, are feasible and individually rational. For $r_z$ sufficiently small the set $B\left((y^i_{k,z})_{i\in\mathcal{N}}, r_z\right)$ is not contained in the closure of the $\eta$-efficient allocations. Since this argument holds for all $z \in \mathbf{D}$ with $\tau(z) \leq T$ we have found an open set, $B = \prod_{z\in\mathbf{D},\tau(z)\leq T} B\left((z^i_{k,z})_{i\in\mathcal{N}}, r_z\right)$, that is SIRF and full-dimensional. For $\max_{z\in\mathbf{D},\tau(z)\leq T} r_z$ sufficiently small, the set $B$ is not contained in the closure of the $\eta$-efficient allocations.

Letting $T \to +\infty$, it follows from the Mackey-continuity of the players' preferences (under our general assumptions, see Bewley, 1972) that, if $(w^i)_{i\in\mathcal{N}}$ is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient (stream of) allocations, then there exists a non-empty, $\tau(\ell^\infty, \ell^1)$-open subset of $\ell^\infty(\mathbf{D})$ that is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient allocations. □

**Proof of Corollary 6.1.** It follows from Theorem 1 and Lemma A.1 that whatever being the number $N$, if players are sufficiently patient, then under (G) and (IA), if $(w^i)_{i\in\mathcal{N}}$ is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient allocations, there exists a non-empty, $\tau(\ell^\infty, \ell^1)$-open subset of $\ell^\infty(\mathbf{D})$ that is not in the $\tau(\ell^\infty, \ell^1)$-closure of the set of $\eta$-efficient allocations and that can nevertheless be approximated by a PBE (in the sense of Theorem 1). □

# References

Acemoglu, D. and Simsek, A. (2010). Moral hazard and efficiency in general equilibrium with anonymous trading. Working Paper 10-8, MIT Department of Economics. Available at SSRN: http://ssrn.com/abstract=1597907.

Amir, R., Sahi, S., Shubik, M., and Yao, S. (1990). A strategic market game with complete markets. *Journal of Economic Theory*, 51(1):126–143.

Araujo, A., Fajardo, J., and Páscoa, M. R. (2005). Endogenous collateral. *Journal of Mathematical Economics*, 41(4-5):439–462.

Arnott, R. and Stiglitz, J. E. (1986). Moral hazard and optimal commodity taxation. *Journal of Public Economics*, 29(1):1–24.

Arnott, R. and Stiglitz, J. E. (1990). The welfare economics of moral hazard. In *H. Louberge (editor) Risk, Information and Insurance: Essays in the Memory of Karl H. Borch*, pages 91–122. Kluwer.

Arnott, R. and Stiglitz, J. E. (1991). Moral hazard and nonmarket institutions: Dysfunctional crowding out of peer monitoring? *The American Economic Review*, 81(1):179–190.

Aumann, R. J. (1976). Agreeing to disagree. *The Annals of Statistics*, 4(6):1236–1239.

Bewley, T. (1972). Existence of equilibria in economies with infinitely many commodities. *Journal of Economic Theory*, 4(3):514–540.

Blackwell, D. (1956). The analog of the minimax theorem for vector payoffs. *Pacific Journal of Mathematics*, 6(1):1–8.

Blume, L. and Easley, D. (2006). If you're so smart, why aren't you rich? belief selection in complete and incomplete markets. *Econometrica*, 74(4):929–966.

Brangewitz, S. (2012). *Coalitional and Strategic Market Games*. PhD thesis, Bielefeld University.

Cao, D. (2011). Collateral shortages, asset price and investment volatility with heterogeneous beliefs. Unpublished.

Debreu, G. (1952). A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences of the United States of America*, 38(10):886–893.

Dubey, P. and Geanakoplos, J. (2003). Monetary equilibrium with missing markets. *Journal of Mathematical Economics*, 39(5-6):585–618.

Duffie, D. and Shafer, W. (1985). Equilibrium in incomplete markets: I : A basic model of generic existence. *Journal of Mathematical Economics*, 14(3):285–300.

Facchinei, F. and Kanzow, C. (2010). Generalized Nash equilibrium problems. *Annals of Operations Research*, 175(1):177–211.

Friedman, M. (1953). *Essays in Positive Economics*. University of Chicago Press.

Fudenberg, D., Levine, D., and Maskin, E. (1994). The folk theorem with imperfect public information. *Econometrica*, 62(5):997–1039.

Geanakoplos, J. and Zame, W. R. (2007). Collateralized asset markets. *UCLA Working Paper*.

Giraud, G. (2003). Strategic market games: an introduction. *Journal of Mathematical Economics*, 39:355–375.

Giraud, G. and Stahn, H. (2003). Efficiency and imperfect competition with incomplete markets. *Journal of Mathematical Economics*, 39(5-6):559 – 583.

Giraud, G. and Weyers, S. (2004). Strategic market games with a finite horizon and incomplete markets. *Economic Theory*, 23:467–491.

González-Díaz, J. and Meléndez-Jiménez, M. A. (2011). On the notion of perfect bayesian equilibrium. *TOP*, pages 1–16.

Greenwald, B. C. and Stiglitz, J. E. (1984). Externalities and economies with imperfect information and incomplete markets. *Quarterly Journal of Economics*, 101(2):229–264.

Harker, P. T. (1991). Generalized Nash games and quasi-variational inequalities. *European Journal of Operational Research*, 54(1):81–94.

Hart, O. D. (1975). On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory*, 11(3):418–443.

Heifetz, A. (2006). The positive foundation of the common prior assumption. *Games and Economic Behavior*, 56(1):105–120.

Holmstrom, B. and Tirole, J. (1997). Financial intermediation, loanable funds, and the real sector. *Econometrica*, 21:265–293.

Kehoe, T. J. and Levine, D. K. (1993). Debt-constrained asset markets. *The Review of Economic Studies*, 60(4):865–888.

Kubler, F. and Schmedders, K. (2003). Stationary equilibria in asset-pricing models with incomplete markets and collateral. *Econometrica*, 71(6):1767–1793.

Magill, M. and Quinzii, M. (1994). Infinite horizon incomplete markets. *Econometrica*, 62(4):853–880.

Martins-Da-Rocha, F. and Vailakis, Y. (2011). Refining not-too-tight solvency constraints. *mimeo*.

Mertens, J. F. (2003). The limit-price mechanism. *Journal of Mathematical Economics*, 39:433–528.

Peck, J., Shell, K., and Spear, S. E. (1992). The market game: Existence and structure of equilibrium. *Journal of Mathematical Economics*, 21:271–299.

Postlewaite, A. and Schmeidler, D. (1978). Approximate efficiency of non-Walrasian Nash equilibria. *Econometrica*, 46(1):127–135.

Prescott, E. C. and Townsend, R. M. (1984a). General competitive analysis in an economy with private information. *International Economic Review*, 25(1):1–20.

Prescott, E. C. and Townsend, R. M. (1984b). Pareto optima and competitive equilibria with adverse selection and moral hazard. *Econometrica*, 52(1):21–46.

Radner, R. (1979). Rational expectations equilibrium: Generic existence and the information revealed by prices. *Econometrica*, 47(3):655–678.

Rahi, R. (1995). Optimal incomplete markets with asymmetric information. *Journal of Economic Theory*, 65(1):171–197.

Rothschild, M. (1974). A two-armed bandit theory of market pricing. *Journal of Economic Theory*, 9(2):185–202.

Rothschild, M. and Stiglitz, J. (1976). Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *The Quarterly Journal of Economics*, 90(4):629–649.

Sandroni, A. (2000). Do markets favor agents able to make accurate predictions? *Econometrica*, 68(6):1303–1341.

Shapley, L. S. and Shubik, M. (1977). Trade using one commodity as a means of payment. *Journal of Political Economy*, 85:937–968.

Spence, M. (1973). Job market signaling. *The Quarterly Journal of Economics*, 87(3):355–374.

Townsend, R. (1979). Optimal contracts and competitive markets with costly state verification. *Journal of Economic Theory*, 21(3):663–691.

Weyers, S. (2004). Convergence to competitive equilibria and elimination of no-trade (in a strategic market game with limit prices). *Journal of Mathematical Economics*, 40(8):903–922.

Wiseman, T. (2012). A partial folk theorem for games with private learning. *Theoretical Economics*, 7(2):217–239.